THE BRAUER COMPLEX AND ITS APPLICATIONS TO THE CHEVALLEY GROUPS

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INTRODUCTION

This thesis is concerned with the determination of the connected centralizers of semi-simple elements in a Chevalley group. To deal with this problem we shall use the recent work of R. Carter [6] and a new tool for the study of algebraic groups - the so called Brauer complex. This complex has been first defined by J. Humphreys [11] in the context of the modular representation theory of the finite Chevalley groups of universal type. Now, in our version, the Brauer complex can be also used for the ordinary representation theory of the finite Chevalley groups of adjoint type. For, Deligne and Lusztig in their fundamental work [9] have constructed for these groups certain families of irreducible complex representations whose degrees can be obtained if we know what subgroups of the finite Chevalley groups are the connected centralizers of semi-simple elements.

The thesis is organized as follows.

In Chapter 1 we introduce notation, definitions and a general background for the theory of Chevalley groups adequate for the rest of the discussion in the thesis. In particular we emphasize the properties of the semi-simple conjugacy classes in the Chevalley groups developed by Springer and Steinberg in the seminar E of [2].

In Chapter 2 we discuss the relation between the semi-simple conjugacy classes of a Chevalley group of universal type and the affine Weyl group. This relation along with a result in [6] will enable us to obtain information about the structure of the connected centralizers of semi-simple elements in any algebraic Chevalley group. This information is given in Theorem 2.11.

In Chapter 3 the Brauer complex is introduced. This complex is used to obtain a geometrical picture of the σ-stable semi-simple conjugacy classes, where σ is the Frobenious map. This is given in Theorem 3.4. A consequence of this theorem is that the number of σ-stable semi-simple classes is q^{ℓ} (ℓ = rank of the group), a well known result.

Chapter 4 begins with a review of Carter's recent work [6]. Then we relate this work with the Brauer complex. This will give the necessary material to determine the orders of the connected centralizers of semi-simple elements in the finite Chevalley groups.

Chapter 5 gives the tables for the orders of the connected centralizers in the finite exceptional groups. A considerable amount of detailed work was involved in the compilation of these tables which has not been included in the thesis. However a number of examples are given to illustrate the sort of calculations involved. These tables together with the results in [7] give a complete list of all the connected centralizers of semi-simple elements for all finite Chevalley groups.

The thesis ends with Chapter 6. In this last chapter we discuss how the Brauer complex can be used to compute the number of semi-simple classes in a finite Chevalley group of universal type whose centralizers are conjugate. From this discussion we can easily determine these numbers for groups of rank ≤ 2, which are already known.

Throughout the thesis we denote by $\mathbb{Z}$, $\mathbb{Z}^{+}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ respectively the integer, the positive integer, the rational, the real and the complex numbers.

To the best of the author's knowledge, those results in this thesis which are not otherwise attributed are original.

CHAPTER 1

Notation and definitions

1.1. The Chevalley groups

Let g be a simple complex Lie algebra of rank ℓ and $H_{\mathbb{C}}$ a Cartan subalgebra in g. We denote by $(,)$ the Killing form of g. This is a non-degenerate form on g. The set of all non-zero roots of g with respect to $H_{\mathbb{C}}$ will be denoted by Φ. For each root r in Φ we define the (unique) element h_r of $H_{\mathbb{C}}$ by $(h,h_r) = \frac{2r(h)}{\langle r,r\rangle}$, for all $h \in H_{\mathbb{C}}$, where $\langle , \rangle$ is the transfer of $(,)$ on $H_{\mathbb{C}}^*$-the dual space of $H_{\mathbb{C}}$. Let Φ^V be the set of all vectors h_r, $r \in \Phi$. We consider the real subspaces $H = H_{\mathbb{R}}$ and $H^* = H_{\mathbb{R}}^*$ of $H_{\mathbb{C}}$ and $H_{\mathbb{C}}^*$ generated by Φ^V and Φ respectively. The restrictions of $(,)$ and $\langle , \rangle$ on H and H^* are symmetric, positive definite bilinear forms and thus H and H^* are Eulidean spaces. The sets Φ and Φ^V are indecomposable and reduced root systems in H^* and H respectively. The set Φ^V is said to be the *dual root system* of Φ. The elements in Φ^V are called coroots. We fix a fundamental basis $\Delta = \{r_1,r_2,\ldots,r_\ell\}$ in Φ. Then the basis $\Delta^V = \{h_i;r_i \in \Delta\}$ of H is a fundamental basis in Φ^V, where h_i is the coroot associated with r_i. In Φ there is a unique root r_o-called the *highest root* of Φ - which has the property such that if $r_o = \sum_{i=1}^{\ell} n_i r_i$ and $r = \sum_1^{\ell} m_i r_i$ is any root in Φ, then $n_i \geq m_i$ for all $i = 1,\ldots,\ell$. We shall call n_i's the coefficients of the highest root. Finally, in Φ there can be only two root lengths and in case we have two distinct lengths, we speak of *long* and *short* roots. In particular, the highest root r_o is a long root. Analogous notions we have for the dual root system $\Phi^\vee$. We consider now the dual bases $\{\lambda_i; i=1,\dots,\ell\}$ and $\{\gamma_i; i=1,\dots,\ell\}$ of $\Delta^\vee$ and Δ respectively. The linear functions λ_i's in H^* are called the *fundamental weights* and the elements γ_i's in H, the *fundamental coweights*. We denote respectively by P, X, Y and Ψ the Z-lattices $\sum_1^\ell Zr_i$, $\sum_1^\ell Z\lambda_i$, $\sum_1^\ell Zh_i$ and $\sum_1^\ell Z\gamma_i$. We shall call these lattices, the *root lattice*, the *weight lattice* the *dual root lattice* and the *dual weight lattice* respectively. The linear functions on H which belong in X will be called *weights*, while the elements in Ψ *coweights*. The quotient groups $\frac{X}{P}$ and $\frac{\Psi}{Y}$ are isomorphic in duality over the group $\mathbb{Q}/Z$. The relation matrix of $\frac{X}{P}$ is the inverse of the Cartan matrix (A_{ij}), where $A_{ij} = r_i(h_j)$ $i,j = 1,2,\dots,\ell$, and the relation matrix of $\frac{\Psi}{Y}$ is the inverse of the transpose of (A_{ij}).

Let now V be a finite dimensional faithful g-module. Then V as an $H_{\mathbb{C}}$-module has a decomposition $V = \bigoplus_\mu V_\mu$, where μ runs over X and $V_\mu = \{v \in V/hv = \mu(h)v$, for all $h \in H_{\mathbb{C}}\}$. The weights μ for which $V_\mu \neq 0$ are called the weights of $H_{\mathbb{C}}$ on V or simply the weights of V.

By definition an *admissible lattice* M of V is a Z-form of a basis of V stable under the action of the elements $\frac{x_r^n}{n!}$ $r \in \Phi$, $n \in Z^+$, where $\{x_r, r \in \Phi, h_i, i=1,\dots,\ell\}$ has been chosen to be a Chevalley basis of g. Notice that there exists a Z-form U_Z of the universal enveloping algebra U of g-called the Kostant form of U-which results the existence of an admissible lattice in V.

Now for each $r \in \Phi$, we have $x_r V_\mu \subseteq V_{\mu+r}$. Thus the action of x_r on V affords a nilpotent endomorphism of V. So we can define the automorphism $\exp x_r = \sum_{n=0}^\infty \frac{x_r^n}{n!}$ on V, which is unipotent, since $\exp x_r - 1$ is nilpotent.

Now let $\mathbb{F}_q$ be a field of q elements, fixed forever, and K its algebraic closure. Let k be any field between $\mathbb{F}_q$ and K. Since M is admissible, $\exp(x_r \otimes u)$ is an automorphism of the Z-module $M \otimes Z[u]$, where u is an indeterminate. Thus $\exp(x_r \otimes u \otimes 1_k)$ is an automorphism of $M \otimes Z[u] \otimes k$. Let t be an element of k. Then we define a homomorphism of $M \otimes Z[u] \otimes k$ into $M \otimes Z[t] \otimes k \simeq M \otimes k$ such that $u \to t$. Through this homomorphism we get an automorphism $x_{r,V}(t) = \exp(x_r \otimes t \otimes 1_k)$ of $V_k = M \otimes k$.

The group $G_V(k)$ generated by all $x_{r,V}(t)(r \in \Phi,\ t \in k)$ is called the *Chevalley group* of type (g,Π,k), where $\Pi = \Pi(V)$ is the Z-lattice generated by the weights of V. Up to isomorphism the groups $G_V(k)$ depend only on the Z-lattice $\Pi(V)$. So it is more reasonable to denote the above group by $G_\Pi(k)$. On the other hand, if we are given a Z-lattice Π between X and P, then there exists a finite dimensional faithful g-module V such that $\Pi = \Pi(V)$. Thus there are as many Chevalley groups of types (g,Π,k) as Z-lattices there are between X and P.

Now every Chevalley group $G_\Pi(K)$ is a connected algebraic group defined over the prime field of K, being generated by its connected subgroups $X_r = \langle x_{r,\Pi}(t); t \in K\rangle$ - called the *root subgroups* - each of which is isomorphic to the additive group K.

A non-abelian connected algebraic group G defined over K which possesses no closed connected normal subgroups is called *simple* (is not simple as abstract group but it can have only finite normal subgroups).

A connected algebraic group G defined over K is called *semi-simple* if it is the (almost) direct product of simple groups. That is $G = G_1 \ldots G_k$, for some $k > 0$, where G_i, $i = 1,\ldots,k$, are simple connected algebraic groups such that each G_i intersects the product of the remaining members $G_j, j \neq i$, in a central (hence finite) subgroup.

Chevalley has shown that every simple connected algebraic group defined over K is isomorphic (as an algebraic group) to an algebraic group of the form $G_\Pi(K)$ and each of the groups $G_\Pi(K)$ is a simple connected algebraic group. This identification of the simple connected groups can be extended to the semi-simple connected groups defined over K. For this we have just to replace the Lie algebra g by a semi-simple one.

Among the Chevalley groups we shall distinguish two particular types of groups. These are the groups $G_X(K)$ and $G_P(K)$; the former is called *simply connected* (or the Chevalley group of universal type) and the latter *adjoint*. We shall denote them by G_{sc} and G_{ad} respectively. The group G_{sc} covers all the groups $G_\Pi(K)$ in the sense that each $G_\Pi(K)$ is isomorphic with the factor group G_{sc}/N, where N is some subgroup of the centre Z of G_{sc}. In particular $G_{ad} \simeq G_{sc}/Z$ is simple as an abstract group. The collection $\{G_\Pi(K); X \supset \Pi \supset P\}$ is called the *isogeny class* of type (g,K). Given a Chevalley group $G = G_\Pi(K)$ and a subfield k of K, then the group $G_\Pi(k)$ is its own derived subgroup and it is the derived subgroup of the group of k-rational points of G, i.e. the group $G \cap GL(M \otimes_Z k)$. If $G = G_{sc}$, then $G_{sc}(k) = G_{sc} \cap GL(M \otimes_Z k)$.

1.2. Tori

In what follows by G we shall denote a simple Chevalley group $G_{\Pi}(K)$.

A *torus* of G is a closed subgroup which is isomorphic to a direct product of s copies of K^*, for some $s > 0$, where K^* is the multiplicative group of K. A *maximal torus* of G is one contained in no other, so that s is equal to ℓ, the *rank* of G.

A standard maximal torus used for the study of G is constructed as follows. Let us denote by $\Lambda(\Pi)$ the group $\mathrm{Hom}_Z(\Pi, K^*)$. Every element χ in $\Lambda(X)$ gives rise to an element of $\Lambda(\Pi)$ by restriction. However, not every element of $\Lambda(\Pi)$ need be the restriction of some element of $\Lambda(X)$. We consider the group $\Lambda'(\Pi)$ of all elements of $\Lambda(\Pi)$ which can be extended to elements of $\Lambda(X)$. For each $r_i \in \Delta$ and $z_i \in K^*$, we define the elements $\chi_{r_i,z_i,\Pi}$ of $\Lambda'(\Pi)$ by $\chi_{r_i,z_i,\Pi}(\mu) = z_i^{\mu(h_i)}$, where $\mu \in \Pi$. Then each element χ of $\Lambda'(\Pi)$ is written uniquely in the form $\prod_1^{\ell}\chi_{r_i,z_i\Pi}$, where $z_i = \chi(\lambda_i)$, $i = 1,2,\ldots,\ell$. Let $\mu_1, \mu_2, \ldots, \mu_n$ be the weights of a fixed basis of the admissible lattice M. Then for each $\chi \in \Lambda'(\Pi)$ we define the diagonal matrix $h(\chi) = \mathrm{diag}(\chi(\mu_1), \ldots, \chi(\mu_n))$. From the Chevalley theory $h(\chi) \in G$, for all $\chi \in \Lambda'(\Pi)$. On the other hand, for each $r_i \in \Delta$, we see that the group generated by $h(\chi_{r_i,z_i,\Pi})$, $z_i \in K^*$, is isomorphic with K^*. Since $\chi = \prod_1^{\ell}\chi_{r_i z_i,\Pi}$, for some $z_i \in K^*$, we also have $h(\chi) = \prod_1^{\ell} h(\chi_{r_i,z_i,\Pi})$ (uniquely expressible). Thus the group $\langle h(\chi);\chi \in \Lambda'(\Pi)\rangle$ is isomorphic with the direct product of ℓ copies of K^*, and so is a maximal torus in G. We shall denote this torus by T_o. We see also that $\Lambda'(\Pi) \simeq T_o$ under the map $h : \chi \to h(\chi)$.

For a subfield k of K, T_o is diagonalizable over k, that is, the group $T_o(k)$ of the k-rational points of T_o is isomorphic with the direct product of ℓ copies of k^*. We say that T_o is a *k-split torus* of G.

Let $X(T_o) = \mathrm{Mor}(T_o, K^*)$ be the group of K-rational characters of T_o. For each $\mu \in \Pi$, we correspond an element ϕ_μ of $X(T_o)$ defined by $\phi_\mu(h(\chi)) = \chi(\mu)$, for all $\chi \in \Lambda'(\Pi)$. This correspondence defines an isomorphism of Π onto $X(T_o)$ and we can identify (and so will do) $X(T_o)$ with Π. In particular, each root r in Φ is in $X(T_o)$ (under the identification) and the root system Φ of g is also called the root system of G with respect to T_o.

A K-rational homomorphism y from K^* into T_o is called *one-parameter subgroup* of T_o. The set of these is denoted by $Y(T_o)$. For any two elements $y_1, y_2 \in Y(T_o)$, one defines the element $y = y_1 + y_2$ by $y(t) = y_1(t)y_2(t)$, $t \in K^*$. Thus $Y(T_o)$ becomes an additive group.

Let ω_i, $i = 1,\ldots,\ell$ be the elements of $X(T_o)$ defined by $\omega_i(t_1,t_2,\ldots,t_\ell) = t_i$. Then ω_i is a basis of $X(T_o)$. For each $\mu \in X(T_o)$ and $y \in Y(T_o)$, the composition map $\mu \circ y.\ K^* \to K^*$ is rational, and so there is an integer n such that $(\mu \circ y)(t) = t^n$, for all $t \in K^*$. This allows one to define a Z-pairing $\langle , \rangle$: $X(T_o) \times Y(T_o) \to Z$ by $\langle \mu, y \rangle = n$, whenever $(\mu \circ y)(t) = t^n$, $t \in K^*$. Under this pairing $X(T_o)$ and $Y(T_o)$ become dual Z-modules. The dual basis of ω_i, $i = 1,\ldots,\ell$ is formed by the elements $y_i, i=1,\ldots,\ell$ which correspond to each $t \in K^*$, the element of T_o whose the i^{th}

coordinate is t and the other ones equal to 1. $Y(T_o)$ will be called the *lattice of one-parameter subgroups*.

The Weyl group.

We have identified the group $X(T_o)$ with Π. This identification can be extended to the real vector spaces H^* and $X(T_o) \otimes_Z \mathbb{R}$. The (finite) group generated by all the reflections ω_r in the hyperplane orthogonal to the root r is called the *Weyl group* of G with respect to T_o. We know that W is isomorphic with $N(T_o)/T_o$, where $N(T_o)$ is the normalizer of T_o in G. We shall fix a set $\{n_\omega;\ \omega \in W\}$ of coset representatives of T_o in $N(T_o)$.

W acts on T_o by $\omega(h(\chi)) = h(\omega(\chi))$, where $\omega(\chi):\mu \to \chi(\omega^{-1}(\mu))$, for all $\mu \in X(T_o)$. Also W acts on $Y(T_o)$ by $(\omega(y))(t) = n_\omega y(t) n_\omega^{-1}$. Further, the Killing form is W-invariant.

The Frobenius endomorphism.

Let σ: $G \to G$ be the map - called Frobenius map - which raises every matrix entry of an element $g \in G$ to its q^{th} power. σ is an automorphism of abstract groups but not an automorphism of algebraic groups. However, σ is a morphism. The group G_σ of the fixed points of G under σ is finite and is just G(q), the group of $\mathbb{F}_q$-rational points of G. A *torus* of G_σ is defined to be the group T_σ of the σ-stable elements of a σ-stable torus T of G, and a maximal torus to be a subgroup obtained in the same way from a maximal torus of G, e.g. T_o is obviously σ-stable and $T_{o\sigma}$ is a maximal torus of G_σ. We notice that the maximal tori of G_σ need not be maximal in the set of tori of G_σ, e.g. if q = 2, then $T_{o\sigma} = \{1\}$. We shall need the following properties of σ.

(1) σ acts on $X(T_o)$ (via its transpose) and on $Y(T_o)$ by $\sigma(\mu) = q\mu$ and $\sigma(y) = qy$, $\mu \in X(T_o)$, $y \in Y(T_o)$. This action is extended to the real vector spaces H^* and H.

(2) σ commutes with the action of W on T_o.

(3) σ leaves stable each root subgroup X_r. In fact we have $\sigma x_{r,\Pi}(t) = x_{r,\Pi}(t^q)$, for all $t \in K$.

(4) For each element g of G there exists an element g' of G such that $g = g'\sigma(g')^{-1}$.

The property (4) is a consequence of the following theorem.

Lang's basic theorem:

Let G be a connected linear algebraic group and σ an endomorphism (in the sense of algebraic groups) of G onto G such that the subgroup G_σ of G of the fixed points of σ is finite. Then the map $f:g \to g\sigma(g)^{-1}$ of G into G is surjective.

We recall that in a connected algebraic group over K (and so in G) any two maximal tori are conjugate. However, this is not so for G_σ, where the situation is described as follows. Let us denote by Ω the set of all maximal tori in G. G acts on Ω by conjugation. Also σ acts on Ω such that $\sigma(T^g) = (\sigma T)^{\sigma(g)}$. Let Ω_σ be the set of all maximal tori stable under σ. For $T \in \Omega_\sigma$ and $g \in G_\sigma$, we have $\sigma(T^g) = (\sigma T)^{\sigma(g)} = T^g$, i.e. $T^g \in \Omega_\sigma$. Thus G_σ acts on Ω_σ. For the set $\Omega_\sigma\backslash G_\sigma$ of the orbits of this action we have the following proposition.

Proposition 1.1. ([2]).

There is a bijection of $\Omega_\sigma\backslash G_\sigma$ onto the set of the conjugacy classes of W. This bijection is given by

$$T^g \to \pi(g^{-1}\sigma(g)),\ \text{where}\ \pi\colon N_G(T_o) \to N_G(T_o)/T_o.$$

We will see in §4.1. that Proposition 1.1 is a special case of a general result in [6].

Twisting

Referring to Lang's basic theorem, for each $\omega \in W$, we can find some $g \in G$ such that $n_\omega = g^{-1}\sigma(g)$. Then we have $T_o^{g^{-1}\sigma(g)} = T_o$ or $T_o^g = T_o^{\sigma(g)}$. Since $\sigma T_o = T_o$ we have $\sigma(T_o{}^g) = (\sigma T_o)^{\sigma(g)} = T_o^{\sigma(g)}$. Thus $T_o{}^g = \sigma(T_o{}^g)$, i.e. $T_o{}^g = T \in \Omega_\sigma$. Conversely, let $T \in \Omega_\sigma$; then there exists some $g \in G$ such that $T = T_o{}^g$. Hence $\sigma(T_o{}^g) = \sigma T = T$ or $(\sigma T_o)^{\sigma(g)} = T_o{}^{\sigma(g)} = T$, i.e. $T_o{}^{\sigma(g)} = T = T_o{}^g$; that is $T_o{}^{g^{-1}\sigma(g)} = T_o$. Therefore $g^{-1}\sigma(g) = n_\omega \in N_G(T_o)$ for some $\omega \in W$. So all the maximal tori in G fixed by σ can be obtained from T_o by "twisting" by some $\omega \in W$. Let T be obtained by T_o by twisting by $\omega \in W$, where $n_\omega = g^{-1}\sigma(g)$. If we identify T with T_o according to the inner automorphism Int g, then the original action of σ on T is equivalent to that of $\omega\circ\sigma$ on T_o. For if $t \in T_o$, and correspondingly $t' \in T$ such that $gtg^{-1} = t'$, then:

$\mathrm{Int}\ g\circ(\omega\circ\sigma)(t) = \mathrm{Int}\ g\circ(\omega\circ\sigma)\circ\mathrm{Int}\ g^{-1}(t') = \mathrm{Int}(gn_\omega\sigma(g)^{-1})(\sigma(t')) = \sigma\circ\mathrm{Int}\ g(t)$. Hence the following diagram commutes:

$$\begin{tikzcd} T_o \arrow[r, "\mathrm{Int}\ g"] \arrow[d, "\omega\circ\sigma"'] & T \arrow[d, "\sigma"] \\ T_o \arrow[r, "\mathrm{Int}\ g"'] & T \end{tikzcd}$$

The above identification is carried out to the character groups $X(T)$ and $X(T_o)$ under the isomorphism Int g^* : $X(T) \to X(T_o)$ defined by Int $g^*(\lambda))(t) = \lambda(\mathrm{Int}g(t))$. Under this identification the new action of σ(via its transpose) on $X(T_o)$ is given by

$$\sigma(\mu) = q\omega(\mu) \text{ , for all } \mu \in X(T_o).$$

We finish this section by stating some well known facts concerning the semi-simple conjugacy classes in G and G_σ.

In general, if A is a group, B a subgroup of A and s an element of A, we denote the conjugacy class of s in A (resp. in B) by $[s]_A$ (resp. by $[s]_B$). We put also $C_A(s)$ (resp. $C_B(s)$) to denote the centralizer of s in A (resp. in B).

We recall that every semi-simple element (i.e. diagonalizable) of G is contained in a maximal torus and every torus, of course, consists of semi-simple elements. Thus every conjugacy class consisting of semi-simple elements can be represented by an element of T_o, as the maximal tori are all conjugate to each other in G. This holds true for G_σ. That is, every σ-stable semi-simple element is contained in a σ-stable maximal torus. In particular, the connected centralizer $C_G(x)^o$ (i.e. the connected component of $C_G(x)$ containing 1) of a σ-stable semi-simple element x is σ-stable and has the same rank as G. The study of this kind of groups will be one of the central purposes in our work.

For the semi-simple conjugacy classes in G we have the following proposition.

<u>Proposition 1.2</u>. ([2])

There is a bijection from the set of the semi-simple conjugacy classes to the set T_o/W of the orbits of the action of W on T_o.

This bijection is given by $[t]_G \to \overline{t}$, where $\overline{t}$ denotes the orbit of the element t of T_o. In fact, $[t]_G \cap T_o = \overline{t}$.

Now the connection of the semi-simple conjugacy classes in G and G_σ is as follows.

Proposition 1.3. ([2]).

(a). Every σ-stable semi-simple conjugacy class in G contains an element fixed by σ.

(b). If $G = G_{sc}$, then every two semi-simple elements in G_σ which are conjugate in G, they are conjugate in G_σ as well. Therefore in this case (using (a)), we have a 1-to-1 correspondence between the σ-stable semi-simple conjugacy classes of G and the ones in G_σ.

Continuing to assume $G = G_{sc}$, we see by Proposition 1.2. and 1.3. that the W-orbits in T_o fixed by σ correspond to semi-simple conjugacy classes also fixed by σ and so to the ones of G_σ. In other words, an element t of T_o is conjugate to an element of G_σ if and only if $\omega \circ \sigma$ fixes t for some $\omega \in W$.

To this end we state the following proposition taken from [2]:

Proposition 1.4.

Let $x \in T_o$. Then the classes of maximal tori fixed by σ represented by tori containing x are those obtained by twisting T_o by some $\omega \in C_W(x)$.

CHAPTER 2

CENTRALIZERS OF SEMI-SIMPLE ELEMENTS IN THE ALGEBRAIC GROUP G.

From now, unless otherwise indicated, G will denote the simply connected group G_{sc} in an isogeny class of Chevalley groups. The reason for this assumption will become clear at the end of this chapter. Thus in this case we have $X \equiv X(T_o)$ and $Y \equiv Y(T_o)$.

In the present chapter we exploit an exact sequence used by G. Lusztig in his work for the ordinary representation theory of G_σ. The analysis of this sequence will allow us to associate to a semi-simple conjugacy class of G_{sc} a certain point of the fundamental region of the action of the affine Weyl group on H. Then this will give a criterion for being a connected reductive subgroup of G the centralizer (being connected in G) of some semi-simple element. This involves the extended Dynkin diagram in a way as Borel-de Siebenthal and Dynkin [10] independently obtained all the connected reductive subgroups of maximal rank in G. Recent work by Carter [6] allows to extend this criterion to any group in the isogeny class of G. This criterion is aimed principally for the first step for finding the orders of the possible connected centralizers of semi-simple elements in G_σ.

It is purposeful to begin first with the affine Weyl group and the extended Dynkin diagram of the root system Φ.

2.1. The affine Weyl group and the extended Dynkin diagram

We consider the Euclidean space $H = Y \otimes \mathbb{R}$. We denote by $H_{r,k}$ ($r \in \Phi, k \in Z$) the hyperplane of H defined by $H_{r,k}=\{h \in H; r(h)=k\}$. Also we denote by $\omega_{r,k}$ the reflection mapping of H onto itself with respect to $H_{r,k}$. Thus

$$\omega_{r,k}(h) = h - r(h)h_r + kh_r.$$

We denote by $d(h)$ for each $h \in H$ the translation mapping defined by

$$d(h)v = v + h,\ v \in H.$$

Then we have $\omega_{r,k} = d(kh_r)\ \omega_r$.
Let us denote by D the group consisting of the translations of the form $d(y), y \in Y$. Now using the obvious relation $\omega d(h)\omega^{-1} = d(\omega(h))$, ($\omega \in W$, $h \in H$), we see that DW is a subgroup of the group of all affine transformations of H onto itself. Obviously we have $D \cap W = \{1\}$, and as $d(\omega(y) \in D$, for all $\omega \in W$ and $y \in Y$ we see that DW is the semi-direct product of D by W.

We shall denote DW by W_α and call it the *affine Weyl group* of Φ. The set of all hyperplanes $H_{r,k}$, ($r \in \Phi$, $k \in Z$) is stable under W_α. In fact we have

$$d(y)\omega\ (H_{r,k}) = H_{\omega(r),k+\omega(r)(y)}.$$

Now the union $\bigcup_{r,k} H_{r,k}$ is a closed subset of H. Hence its complement $H - \bigcup_{r,k} H_{r,k}$ is an open subset of H. Any connected component C of $H - \bigcup_{r,k} H_{r,k}$ is an open simplex, called alcove. One of them is the open simplex

$$C_o = \{h \in H / r_i(h) > 0,\ i = 1,2,\dots,\ell,\ r_o(h) < 1\}.$$

Let us denote by I_o the set $\{0,1,2,\dots,\ell\}$. For each $i \neq 0$, $i \in I_o$, we put $H_i = H_{r_i,0}$ and $H_o = H_{r_o,1}$. The hyperplanes H_i, $i \in I_o$, are called the walls of C_o. For a proper subset J of I_o, we let $H_J = \bigcap_{j\in J} H_j$. We denote by F_J the set of points h in H satisfying $r_j(h) = 0$ for $0 \neq j \in J$, $r_i(h) > 0$, for $0 \neq i \notin J$ and $r_o(h) = 1$ if $0 \in J$, $r_o(h) < 1$ if $0 \notin J$. Then F_J is an open simplex in the affine space H_J with vertices the points $\frac{1}{n_i}\gamma_i$, $i \notin J$, where n_i is the i^{th} coefficient of r_o and $\gamma_o = 0$. We see that $F_\emptyset = C_o$ and $\overline{C}_o = \bigcup_{J \neq I_o} F_J$ - a disjoint union of simplices - where $\overline{C}_o$ denotes the closure of C_o. The simplices F_J, $J \subsetneq I_o$ are called the *faces* of C_o and the subsets J the *types* of the faces.

With the above notation, W_α is characterised by the following proposition.

<u>Proposition 2.1.</u>([3]).

(a) W_α is a Coxeter group on generators $\omega_{r_o,1}, \omega_{r_1}, \ldots, \omega_{r_\ell}$ which are precisely the reflections in the walls of C_o.

(b) W_α acts simply - transitively on the set of all alcoves. The closed simplex $\overline{C}_o$ is taken to be the fundamental region of the action of W_α on H. Thus the collection $\{\omega(\overline{C}_o);\ \omega \in W_\alpha\}$ of closed simplices forms a tessellation of H, which is parametrized by W_α itself.

Let ω be an element of W_α. Then the closed simplex $\omega(\overline{C}_o)$ also can be chosen as the fundamental region of W_α, with respect to which the Coxeter generators of W_α are the elements $\omega\omega_{r_i}\omega^{-1}$, $i = 1,2,\ldots,\ell$ and $\omega\omega_{r_o,1}\omega^{-1}$. Using this fact, one can define an action of W_α on the set of all pairs (C,ρ) by $\omega\cdot(C,\rho) = (\omega(C),\omega\rho\omega^{-1})$, where C is a closed simplex obtained from $\overline{C}_o$ under W_α, and ρ a Coxeter generator of W_α with respect to C. Now each W_α-orbit of this set of pairs intersects each of the $\{C\} \times \Omega(C)$ in exactly one pair, where $\Omega(C)$ denotes the set of Coxeter generators of W_α with respect to C. Since $\{C\} \times \Omega(C)$ has $\ell + 1$ elements we have $\ell + 1$ orbits. So the set I_o can also be identified with the set of the above $\ell + 1$ orbits. Thus each element of W_α induces the identity map on I_o, since fixes each orbit.

It is clear now that the action of W_α on H does not affect the types of the faces, i.e. the face $\omega(F_J)$ of $\omega(C_o)$, for all $\omega \in W_\alpha$, is of type J as W_α acts trivially on I_o.

Let us consider now the group $\tilde{W}$ of all affine transformations of the form $d(\gamma)\omega$, $(\gamma \in \Psi, \omega \in W)$. We know that the elements of Ψ are vertices of alcoves (called special points). This means $\tilde{W}$ permutes the alcoves. W_α is a subgroup of $\tilde{W}$. In fact, $\omega(H_{r,k}) = H_{r',m}$, $(\omega \in \tilde{W},\ r,r' \in \Phi, k,m \in Z)$ implies $\omega\omega_{r,k}\omega^{-1} = \omega_{r',m}$, i.e. W_α is normal in $\tilde{W}$. Let $\mathcal{D}$ be the subgroup of $\tilde{W}$ fixing $\overline{C}_o$. $\mathcal{D}$ is called the *fundamental group* of Φ and is characterized by the following properties, (see [3]):

(a) $\mathcal{D}$ consists of the elements $d(\gamma_j)\sigma_j$, where j runs over the set $\{j \in I_o / n_j = 1\}$ and $\sigma_j = \omega_j\omega_o$, ω_o being the unique element of maximal length in W and ω_j the one in the subgroup generated by all ω_{r_i}, $i \neq j$.

(b) $\mathcal{D}$ acts by conjugation on W_α and permutes its generators $\omega_{r_o,1}, \omega_{r_1}, \ldots \omega_{r_\ell}$ in such a way that $r_i(h_k) = r_{\rho_j(i)}(h_{\rho_j(k)})$, $i,k \in I_o$, where ρ_j is the permutation on I_o induced by the element $d(\gamma_j)\sigma_j$.

Thus $\tilde{W}$ is the semi-direct product of W_α by $\mathcal{D}$ and we have the isomorphism $\mathcal{D} \to \Psi/Y$ given by $d(\gamma_j)\sigma_j \to \gamma_j + Y$.

Let now $\tilde{\Delta}$ be the set $\{-r_o, r_1, \ldots, r_\ell\}$. We call *extended Dynkin diagram* of Φ the graph defined as follows:

(a) the set I_o is the set of vertices

(b) the vertex i is joined with the vertex j, for $j \neq i$, by $r_i(h_j)r_j(h_i)$ edges.

(c) whenever two vertices i and j are joined by more than one edge then we add an arrow pointing to the vertex which corresponds to the shorter root of the two roots r_i and r_j (or $-r_j$ if $j = 0$). Without abuse of language, we shall say "the extended Dynkin diagram $\tilde{\Delta}$" of Φ.

Let Δ_J be the subset of $\tilde{\Delta}$ consisting of the roots r_j, $j \in J$, where J is a proper subset of I_o. Let Φ_J be the subroot system of Φ generated by Δ_J and W_J the Weyl subgroup of W generated by all ω_{r_j}, $r_j \in \Delta_J$. The graph obtained by omitting the vertices $i \in I_o - J$ of $\tilde{\Delta}$ is called the *Dynkin diagram* Δ_J of Φ_J. Notice that there may be more subroot systems in Φ of different types from the Φ_J's, but here we shall confine ourselves only to Φ_J's, $J \subsetneq I_o$.

Let τ be a permutation on a subset J of I_o. We say that τ is a *symmetry* of Δ_J if $r_i(h_k) = r_{\tau(i)}(h_{\tau(k)})$ for all $i,k \in J$. e.g. each element of $\mathcal{D}$ other than the identity induces a distinct symmetry on $\tilde{\Delta}$, whereas each element of W_α induces the trivial one. This kind of symmetries combined with the Frobenius endomorphism give rise to the twisted (or Steinberg) groups which we will meet in Chapter 4.

Finally, for $\Delta_J \subsetneq \tilde{\Delta}$, $\mathrm{Aut}_w(\Delta_J)$ denotes the group of symmetries on Δ_J induced by the elements of W. To find the orders of the possible centralizers of semi-simple elements in G_σ we shall need the following proposition concerning the structure of the normalizer Ω_J of Δ_J in W.

Proposition 2.2. ([5])

(a) Ω_J is isomorphic to $N_W(W_J)/_{W_J}$

(b) W_J is a normal subgroup of Ω_J and we have

$$\Omega_{J/W_J^\perp} \simeq N_W(W_J)_{/W_J} \times W_J^\perp \simeq \mathrm{Aut}_W(\Delta_J),$$

where $W_J^\perp$ is the Weyl group of the orthogonal to Φ_J subroot system $\Phi_J^\perp$.

We shall see later in Chapter 5, as a result of our calculations, that $W_J^\perp$ has always a complement in Ω_J (this has been verified for the classical groups in [7]).

2.2. The p'-rational points in $\overline{C}_o$.

Let us denote by $\mathbb{Q}_{p'}$ the local ring of the rational numbers at the prime p, i.e. $\mathbb{Q}_{p'} = \{p^s \frac{n}{m} / s \in \mathbb{N}, m,n \in Z,\ p\nmid m,n\}$. The mapping $p^s \frac{n}{m} \to e^{2\pi i p^s \frac{n}{m}}$ is a homomorphism of the additive group $\mathbb{Q}_{p'}$ onto the set of all m^{th}-roots of unity, for all m, $(m,p) = 1$. The kernel of this homomorphism is the group Z of integers. Thus we have the exact sequence

$$O \to Z \to \mathbb{Q}_{p'} \to \mathbb{Q}_{p'/Z} \to O.$$

Therefore, also

(1) $\quad O \to Y \to Y \otimes \mathbb{Q}_{p'} \to Y \otimes (\mathbb{Q}_{p'/Z}) \to O$

is an exact sequence of Z-modules (as Y is flat and $Y \otimes Z \simeq Y$). Now as K is algebraically closed field of characteristic $p > 1$, we have $K^* = \bigcup_{e \in \mathbb{N}} \mathbb{F}^*_{q^e}$ and $K^* \simeq \mathbb{Q}_{p'/Z}$.

The lemma which follows will give a parametrization of the semi-simple conjugacy classes of G by certain points in the fundamental region $\overline{C}_o$ of W_α. This parametrization will be of great use in the rest of our discussion in this chapter.

Lemma 2.3.

Let ϕ be a homomorphism of $Q_{p'}$ onto K^*. Then there exists a surjective homomorphism χ of $Y \otimes Q_{p'}$ onto $\Lambda(X) = \mathrm{Hom}(X,K^*)$ given by $\Sigma y_i \otimes \pi_i \to \Pi\chi_{y_i\otimes\phi(\pi_i)}$, where $\chi_{y_i\otimes\phi(\pi_i)}(\lambda) = \phi(\pi_i)^{\lambda(y_i)}$, for all $\lambda \in X$. The kernel of χ is $Y = Y\otimes Z$

Proof:

We consider the map $Y \times Q_{p'} \to \mathrm{Hom}(X,Q_{p'}/Z)$ given by $(y,\pi) \to \chi_{(y,\pi)}$, where $\chi_{(y,\pi)}(\lambda) = \lambda(y)\pi \bmod Z$. Since we obviously have $\chi_{(y_1+y_2,\pi)} = \chi_{(y_1,\pi)}\,\chi_{(y_2,\pi)}$, $\chi_{(y,\pi_1+\pi_2)} = \chi_{(y,\pi_1)}\,\chi_{(y,\pi_2)}$, the above map induces a unique homomorphism of $Y \otimes Q_{p'}$ onto $\mathrm{Hom}(X,Q_{p'}/Z)$ given by $\Sigma y_i \otimes \pi_i \to \chi_{\Sigma y_i\otimes\pi_i} \bmod Z = \Pi\chi_{y_i\otimes\pi_i} \bmod Z$; where $\Pi\chi_{y_i\otimes\pi_i} \bmod Z(\lambda) = \Sigma\lambda(y_i)\pi_i \bmod Z$, for all $\lambda \in X$. Now as $\Sigma\lambda(y_i)\pi_i \in Z$, for all $\lambda \in X$ if and only if $\Sigma y_i \otimes \pi_i \in Y \otimes 1 \equiv Y$ and since we already know from (1) that $\mathrm{Hom}(X,Q_{p'}/Z) \simeq Y \otimes (Q_{p'}/Z) \simeq Y \otimes Q_{p'}/Y$ the above map is surjective. Therefore, under ϕ we have the surjective homomorphism $Y \otimes Q_{p'} \to \mathrm{Hom}(X,K^*)$ given as required: Now taking into account the isomorphism $h:\Lambda(X) \to T_o$, we have the following Corollary.

Corollary 2.4.

(a). Every element of T_o is of the form $h(\chi_y)$ for some $y \in Y \otimes Q_{p'}$.

(b) The semi-simple conjugacy classes of G correspond naturally to the W_α-orbits in $Y \otimes Q_{p'}$.

Proof:

(a) Straightforward from the lemma.

(b) Let $h(\chi_{y_1})$, $h(\chi_{y_2})$ be two elements of T_o for some $y_1, y_2 \in Y \otimes Q_{p'}$. By 1.2 we know that $h(\chi_{y_1})$ and $h(\chi_{y_2})$ are G-conjugate if and only if $\omega(\chi_{y_1}) = \chi_{y_2}$, for some $\omega \in W$. Let $y_1 = \Sigma y_i \otimes \pi_i$, $y_i \in Y, \pi_i \in Q_{p'}$. Then we have $\omega(\chi_{y_1})(\lambda) = \Pi\phi(\pi_i)^{\omega^{-1}(\lambda)(y_i)} = \Pi\phi(\pi_i)^{\lambda(\omega(y_i))} = \chi_{\omega(y_1)}(\lambda)$, for all $\lambda \in X$. Thus $h(\chi_{y_1})$ is G-conjugate to $h(\chi_{y_2})$ if and only if $\chi_{\omega(y_1)} = \chi_{y_2}$, i.e. $\chi_{\omega(y_1)-y_2} = 1$. That is, $\omega(y_1) - y_2 \in Y$ which means that y_1 is W_α-conjugate to y_2. Now as every semi-simple element of G is conjugate to some element of T_o we get the requirement.

We represent the W_α-orbits of $Y \otimes Q_{p'}$ by the points in $Y \otimes Q_{p'} \cap \overline{C}_o$ and we call these points the *p'-rational points* in $\overline{C}_o$. In particular, the p'-rational points in $\overline{C}_o$ which lie in C_o will be called *regular* p'-rational points. With this terminology we can restate 2.4(b) in a form as follows.

Corollary 2.4.(b')

The semi-simple conjugacy classes in G are parametrized by the p'-rational points in $\overline{C}_o$.

Having established this parametrization, we are in a position now to study the structure of the centralizers of semi-simple elements by means of the p'-rational points. This we do in the next section.

2.3. The main theorem on Centralizers.

Let us first introduce some further notation which we shall use in the present section.
Given $\chi \in \Lambda(X)$, we denote by Φ_χ the set of roots $r \in \Phi$ annihilated by χ. Thus $\Phi_\chi = \{r \in \Phi; \chi(r) = 1\} = \{r \in \Phi; r(t) = 1\}$ where $t = h(\chi)$ - the element of T_o corresponding to χ. Obviously, Φ_χ is a subroot system of Φ. Let W_χ denote the Weyl subgroup of W generated by the reflections ω_r, $r \in \Phi_\chi$. If J is a proper subset of I_o, then $W_{\alpha,J}$ will denote the subgroup of the affine Weyl group W_α generated by all ω_{r_j}, $j \in J$ and $\omega_{r_o,1}$ if $0 \in J$. Finally, for a point $y \in \overline{C}_o$ we denote by $W_{\alpha,y}$ the stabilizer of y in W_α.
With the above notation we have the following lemma.

Lemma 2.5.

For $J \subsetneq I_o$, the group $W_{\alpha,J}$ is finite and in fact $W_{\alpha,J} = W_{\alpha,y}$, for some $y \in \overline{C}_o$.

Proof:

The only element in the group of translations D which fixes a point in H is the identity. On the other hand, since J is a proper subset of I_o, the set $\bigcap_{j \in J} H_j$ is not empty. Thus $D \cap W_{\alpha,J} = \{1\}$ So the map $W_\alpha \to W$, $d(y).\omega \to \omega$ is injective on $W_{\alpha,J}$. This proves the first part of the lemma. For the equality $W_{\alpha,J} = W_{\alpha,y}$ see N. Bourbaki [3] p.p. 75 assertion (I).

Having in mind Corollary 2.4 we can now describe, by means of the p'-rational points the centralizer of an element of T_o in W. This is given in the following proposition.

Proposition 2.6.

Let $t_o \doteq h(\chi_{y_o})$ be an element of T_o as in Corollary 2.4, where y_o is a p'-rational point in $\overline{C}_o$. Then the stabilizer W_{α,y_o} -

of y_o is isomorphic with the group $W_{\chi_{y_o}} = \langle \omega_r; r(t_o) = 1\rangle$ - the centralizer of t_o in W.

Proof:

We have the exact sequence

$$1 \longrightarrow Y \longrightarrow W_\alpha \xrightarrow{f} W \longrightarrow 1$$

where $f: d(y)\circ\omega \to \omega$, for $y \in Y$. f is injective on every finite subgroup of W_α. We prove that $f(W_{\alpha,y_o}) = W_{\chi_{y_o}}$. In fact, let $r \in \Phi_{\chi_{y_o}}$. Then $\chi_{y_o}(r) = \Pi\phi(\pi_i)^{r(y_i)} = 1$, where $y_o = \Sigma y_i \otimes \pi_i$, $(y_i \in Y, \pi_i \in Q_{p'})$. Therefore $r(y) \in Z$ which implies that $d(r(y)h_r)\omega_r \in W_{\alpha,y_o}$. Thus $\omega_r \in f(W_{\alpha,y_o})$ for all $r \in \Phi_{\chi_{y_o}}$, which means that $W_{\chi_{y_o}} \subseteq f(W_{\alpha,y_o})$. Conversely, let $\omega \in f(W_{\alpha,y_o})$. Then for some $y \in Y$ one has $d(y)\,\omega \in W_{\alpha,y_o}$. Let us write $d(y)\,\omega$ as a product of reflections relative to the walls of $\overline{C}_o$ passing through y_o: (*) $d(y)\,\omega = \omega_{r_{i_1}} \cdots \omega_{r_{i_j}}$, $i_j \in I_o$. Thus in (*) $i_j \neq 0$ if and only if $r_{i_j}(y_o) = 0$ and $i_j = 0$ if and only if $r_o(y_o) = 1$. Therefore in both cases we have $\chi_{y_o}(r_{ij}) = \Pi\phi(\pi_k)^{r_{ij}(y_k)} = 1$, where $y = \Sigma y_k \otimes \pi_k$. Hence $r_{i_j} \in \Phi_{\chi_{y_o}}$. Thus we have $f(\omega_{r_{i_j}}) \in W_{\chi_{y_o}}$, for $j = 1,\ldots,s$. This shows that $\omega \in W_{\chi_{y_o}}$ and so $f(W_{\alpha,y_o}) \subseteq W_{\chi_{y_o}}$. The first part of the proof shows that $W_{\alpha,y_o} \simeq W_{\alpha_{y_o}}$ as required.

We apply now the above proposition to the centralizers of semi-simple elements in G.

A connected algebraic group C is called reductive if it can be decomposed C = MS with M a semi-simple group and S a central torus. In this decomposition S is uniquely determined as the identity component of the centre of C and M is the derived group of C or else as the largest semi-simple subgroup. The following theorem is due to Steinberg [2].

Theorem 2.7.

In any semi-simple simply-connected group G the centralizer of every semi-simple element is a connected reductive subgroup. Moreover, if t is an element of a maximal torus T, then $C_G(t) = \langle T, X_r / r(t) = 1 \rangle$, where X_r are the root subgroups with respect to the torus T.

We observe that $C_G(t)$ and G have the same rank since $T \subseteq C_G(t)$. Now by 2.6 and 2.7 we have the following Corollary.

Corollary 2.8.

Let t be a semi-simple element in G. Then the centralizer $C_G(t)$ of t is the (unique) maximal torus T containing t (i.e. t is regular) if and only if the p'-rational point y_o which corresponds to the conjugacy class $[t]_G$ is regular.

Now we have the following Proposition which characterizes the centralizers of semi-simple elements in G.

Proposition 2.9.

Except for a finite number of primes, a connected reductive subgroup G_1 of maximal rank in G is the connected centralizer of a semi-simple element of G if and only if some proper subset of the roots in $\tilde{\Delta}$ is equivalent under W to a fundamental basis of the root system of G_1.

Proof:

Let $G_1 = C_G(s)$, s being a semi-simple element in G. We may assume that G_1 contains the torus T_o, since otherwise if T is a maximal torus in G_1 we can identify under an inner automorphism of G, the torus T with T_o and so the root system $\Phi(T)$ with $\Phi = \Phi(T_o)$. Let x_o be the p'-rational point in $\overline{C}_o \cap Y \otimes \mathbb{Q}_p$, which corresponds to the conjugacy class $[s]_G$. Then the element $t_o = h(\chi_{x_o})$ is the element of T_o corresponding to the element χ_{x_o} of $\mathrm{Hom}(X,K^*)$ under the isomorphism $h:\mathrm{Hom}(X,K^*) \to T_o$. We have seen that the root system of the centralizer $C_G(t_o)$ is generated by the roots in $\tilde{\Delta}$ which define the hyperplane on which x_o lies. Since $t_o \in [s]_G$, G_1 is conjugate to $C_G(t_o)$ and an element which conjugate them may be taken to be in $N_G(T_o)$. For the root subgroups we have $n_\omega X_r n_\omega^{-1} = X_{\omega(r)}$, for all $r \in \Phi$ and $\omega \in W$. Thus in our case, this says that the root system of G_1 is W-conjugate to that of $C_G(t_o)$ (by our assumption, at the beginning of the proof, the W-conjugation may be followed by an isomorphism of root systems induced by an inner automorphism of G). Thus, some fundamental system of roots of G_1 is W-conjugate to a proper subset of $\tilde{\Delta}$.

Conversely, suppose that the root system Φ_1 of a connected reductive subgroup G_1 of maximal rank in G is W-conjugate to some subroot system of Φ generated by a proper subset of $\tilde{\Delta}$. We want to show that G_1 is the centralizer of some semi-simple element in G. With what we have seen before, this is equivalent to show the following. If F is the set of all faces F_J of $\overline{C}_o$ such that Δ_J is W-conjugate to a fundamental basis of Φ_1, then for some J, F_J contains a p'-rational point (except for a certain finite number of primes). By definition, a point x lies on the face F_J if

(1) $x = \sum_{i\in I-J} s_i\gamma_i$, $s_i > 0$, $\forall\, i \in I-J$ and

(2)
$$\sum_{i\in I-J} s_i n_i < 1, \text{ if } 0 \notin J$$
$$\sum_{i\in I-J} s_i n_i = 1, \text{ if } 0 \in J,$$

where $I = I_0-\{0\}$, and the n_i's are the coefficients of the highest root with respect to Δ.

Writing x in terms of the coroots we have from (1):

(1)' $x = \sum_{j=1}^{\ell} (\frac{1}{d} \sum_{i\in I-J} s_i d_{ji}) h_j$, where d is the determinant of Cartan matrix and d_{ji} are the (j,i) entries of the transpose matrix of the inverse of the Cartan matrix. Thus x is a p'-rational point if and only if the coefficients $x_i = \frac{1}{d} \sum_{i\in I-J} s_i d_{ji}$ in (1)' are in $Q_{p'}$. On the other hand, we have to consider only rational solutions in order that the x_i's are in $Q_{p'}$. Each rational solution of (2) certainly will define a finite set of primes, that is, those primes p for which the x_i's are not in $Q_{p'}$. So we have to exclude only the primes which are divisors of the greatest common divisor of the least common multiples of the denominators of the x_i's of each rational solution of (2). Thus for each F_J of F we obtain a finite set of primes, say, Q_J. Finally, considering the W-conjugation among the subsets of $\tilde{\Delta}$ which are defined by the elements of F, we must exclude only the primes which are in

$\bigcap_{F_J\in F} Q_J$.

The above Proposition has been already verified in Carter's paper [7], for the classical groups. This paper also gives the primes which have to be excluded for each F in the classical groups. We give below the table of these primes referred in the proposition for each type of the exceptional groups. We see from this table and Carter's paper that these primes are the bad primes for each type of groups. By a bad prime we mean here, a prime number which is a divisor of some coefficient of the highest root.

Table of excluded primes.

Type of G.	Type of centralizers	excluded primes.
G_2	$A_1 + \tilde{A}_1$	2
	A_2	3
	All the other types of Δ_J's, $J \subsetneq I_o$	none
F_4	$2A_1$	2
	$2A_1 + \tilde{A}_1$	2
	$B_2 + A_1$	2
	A_3	2
	$A_2 + \tilde{A}_2$	3
	$A_3 + A_1$	2
	$C_3 + A_1$	2
	B_4	2
	All the other types of Δ_J's, $J \subsetneq I_o$	none

(continued)

Type of G	Type of centralizers	excluded primes
E_6	$4A_1$	2
	$A_3 + 2A_1$	2
	$3A_2$	3
	$A_5 + A_1$	2
	All the other types of Δ_J's, $J \subsetneq I_o$	none
E_7	$[4A_1]'$	2
	$5A_1$	2
	$[A_3 + 2A_1]'$	2
	$3A_2$	2
	$A_3 + 3A_1$	2
	$[A_5 + A_1]'$	2
	$2A_3$	2
	$2A_3 + A_1$	2
	$A_5 + A_2$	3
	$D_4 + 2A_1$	2
	A_7	2
	$D_6 + A_1$	2
	All the other types of Δ_J's, $J \subsetneq I_o$.	none
E_8	$5A_1$	2
	$A_2 + 4A_1$	2
	$3A_2$	3
	$A_3 + 3A_1$	2
	$D_4 + 2A_1$	2
	$3A_2 + A_1$	3

(continued)

Type of G	Type of centralizers	excluded primes
E_8	$A_3 + A_2 + 2A_1$	2
	$2A_3 + A_1$	2
	$A_5 + 2A_1$	2
	$A_5 + A_2$	3
	$[A_7]''$	2
	$D_4 + A_3$	2
	$D_5 + 2A_1$	2
	$D_6 + A_1$	2
	$A_5 + A_2 + A_1$	2 and 3
	$2A_4$	5
	$A_7 + A_1$	2
	A_8	3
	$D_5 + A_3$	2
	D_8	2
	$E_6 + A_2$	3
	$E_7 + A_1$	2
	All the other types of Δ_J's, $J \subsetneq I_o$	none

We state now the following proposition due to R. Carter [6] by means of which Proposition 2.9 holds true for any Chevalley group.

<u>Proposition 2.10.</u> [6]

Let G, $\tilde{G}$ be isogenous Chevalley groups with root system Φ and G_1, $\tilde{G}_1$ be the connected reductive subgroups of G, $\tilde{G}$ respectively with root system Φ_1. Then G_1 is the connected centralizer of some semi-simple element of G if and only if $\tilde{G}_1$ is the connected centralizer of some semi-simple element of $\tilde{G}$.

Now 2.9 and 2.10 give the following theorem.

Theorem 2.11.

If the characteristic of the field K is good, then a connected reductive subgroup G_1 of a Chevalley group G is the connected centralizer of some semi-simple element of G if and only if some proper subset of the roots in $\tilde{\Delta}$ is equivalent under W to a fundamental basis of the root system of G_1.

For a proper subset J of I_o we denote by C_J the set of all connected centralizers of semi-simple elements of G which are G-conjugate to the connected centralizers whose root system is Φ_J. With this notation the following Corollary is an immediate consequence of Theorem 2.11.

Corollary 2.12.

Every connected centralizer of a semi-simple element in a Chevalley group G is in some C_J, $J \subsetneq I_o$.

This Corollary will be needed in Chapter 4 for the first step to parametrize the families of the "semi'simple" irreducible complex representations of $G_{ad,\sigma}$ constructed in [9].

CHAPTER 3

THE BRAUER COMPLEX

We have seen in 1.3. that there is a 1-1 correspondence between the σ-stable semisimple conjugacy classes of G and the ones of G_σ. So by 2.4(b') we obtain a certain finite number of points in $\overline{C}_o$ which parametrize the above classes. Thus it is natural to look for properties which characterize these points, e.g. we could ask how are these points distributed in $\overline{C}_o$ or even what are their coordinates in terms of q. In the present chapter we introduce the main tool to find these properties which we use systematically in the next part of the thesis concerned with the semi-simple classes in G_σ. This is an ℓ-dimensional complex in H (ℓ = dim H). As an abstract geometrical object this complex is the same as the Brauer complex which has been suggested by Humphreys in [11], for the solution of the "decomposition" problem of the modular representation theory of G_σ. The name it bears comes from the Brauer tree of $A_1(q)$ (in Humphreys version).

I take here the opportunity to thank my supervisor Professor R. Carter who stimulated me to use the Brauer complex in order to find the degrees of the "semi-simple" characters of the finite Chevalley groups.

3.1. Definition of the Brauer complex

For a positive integer n we consider the group of translations $\frac{1}{q^n}D$ which we denote D_n. We form the affine reflection group $D_nW = W_{\alpha,n}$.

The groups $W_{\alpha,n}$, $n \in Z^+$, have the same properties as the affine Weyl group $W_{\alpha,0} = W_\alpha$. We enumerate some of them and we introduce the notation which we shall use for the simplices, hyperplanes etc. corresponding to these groups.

(a) $W_{\alpha,n}$ is a Coxeter group on generators $\omega_{r_1},\ldots,\omega_{r_\ell}$ and $\omega_{r_0,\frac{1}{q^n}} = d(\frac{1}{q^n}h_{r_0})\omega_{r_0}$.

(b) The fundamental region of the action of $W_{\alpha,n}$ on H is the closed simplex

$$\overline{\mathcal{A}}_0^n = \frac{1}{q^n}\,\overline{C}_0 = \{h \in H/r_i(h) \geq 0,\ r_0(h) \leq \frac{1}{q^n},\ i = 1,\ldots,\ell\}$$

Let $d(\frac{1}{q^n}y)\omega$ be an element of $W_{\alpha,n}$. Then we shall denote by $\mathcal{A}^n_{\omega,y}$ the open simplex

$$\{h\in H/\omega(r_i)(h) > \frac{1}{q^n}\omega(r_i)(y),\omega(r_0)(h) < \frac{1}{q^n}(\omega(r_0)(y) + 1),\ i = 1,\ldots,\ell\}$$

If $\omega' \in W$, $y' \in Y$ and $m \in \mathbb{N}$, then the element ω' (resp. y') will be called the relative orientation (resp. position) of $\mathcal{A}^{mn}_{\omega'\omega,q^my+\omega(y')}$ with respect to $\mathcal{A}^n_{\omega,y}$.

The open simplices $\mathcal{A}^1_{\omega,y}$, $(\omega \in W,\ y \in Y)$ will be denoted by $\mathcal{A}_{\omega,y}$ and called the (ω,y)-alcoves. In particular the simplex $\mathcal{A}^1_0$ will be denoted by $\mathcal{A}_0$.

(c) The collection $\{\mathcal{Q}^n_{\omega,y}/\omega \in W,\ y \in Y\}$ forms a tesselation of H parametrized by $W_{\alpha,n}$ itself.

Let Σ_n denote the set of hyperplanes $H_{r,\frac{z}{q^n}}$, $(r \in \Phi,\ z \in Z)$.

Then we have

(d) $\Sigma_n = \{\omega(H_i),\ \omega(H_{r_o,\frac{1}{q^n}})/\omega \in W_{\alpha,n}\}$ and of course, $\Sigma_n \subset \Sigma_{n+1}$ as well as $W_{\alpha,n} \subset W_{\alpha,n+1}$ for all $n \in \mathbb{N}$.

The faces of $\mathcal{Q}^n_o$ are the open simplices $\frac{1}{q^n} F_J$, $J \subsetneq I_o$, denoted by F^n_J, e.g. F^o_J is the face F_J of C_o of type J. Also here we say that the face F^n_J is of type J. The face of type J of $\mathcal{Q}^n_{\omega,y}$ will be denoted by $F^n_{J,\omega,y}$. As for the affine Weyl group we have the following property for $W_{\alpha,n}$.

(e) The action of $W_{\alpha,n}$ on H does not affect the types of the faces of $\mathcal{Q}^n_o$, i.e. the face $\omega(F^n_J)$ of $\omega(\mathcal{Q}^n_o)$, for all $\omega \in W_{\alpha,n}$, is of type J.

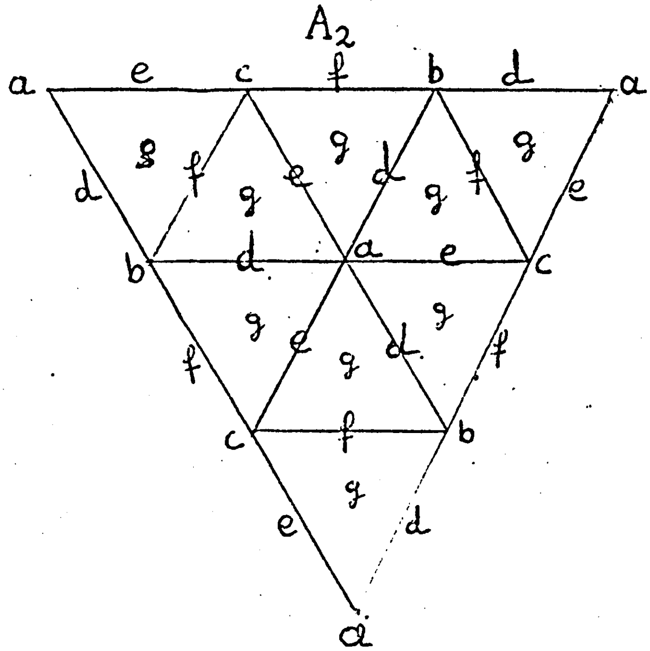
As $\overline{C}_o$ is an ℓ-dimensional simplex we see (because of volume reasons) that it can be filled up by exactly q^ℓ (ω,y)-alcoves. We make now the following definition for the Brauer complex.

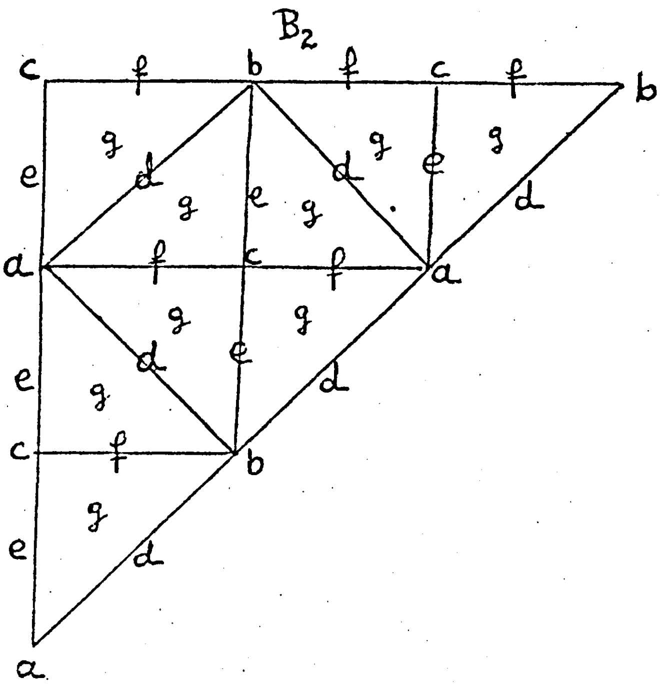
<u>Definition 3.1.</u>

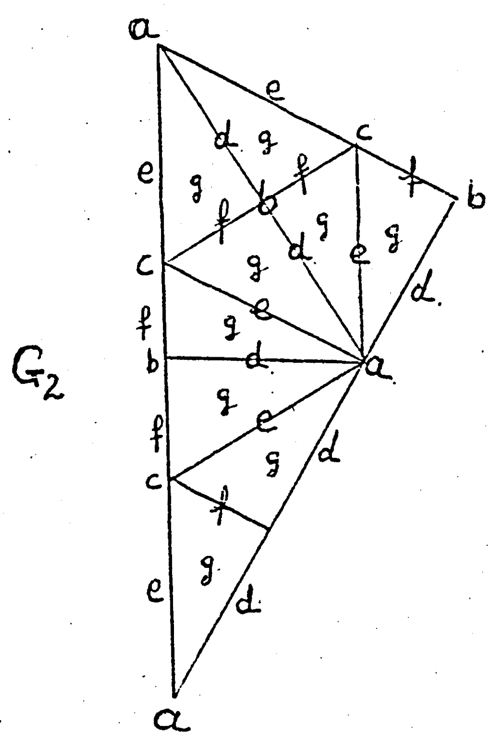
We call Brauer complex and denote it by B_o, the set of all faces of (ω,y)-alcoves which are in $\overline{C}_o$.

So we can speak of (ω,y)-alcoves of the Brauer complex. These are the ℓ-dimensional faces of B_o. We draw below the Brauer complex

B_o for the types A_2, B_2 and G_2, where we show how the various faces are distributed in $\overline{C}_o$ for $q = 3$. a, b and c denote the faces of dimension 0, d, e and f the faces of dimension 1 and g the face of dimension 2.







3.2. The σ-invariant points and their properties

We have seen that every semi-simple class of G gives rise to a p'-rational point in $\overline{C}_o$. In particular every semi-simple class of G gives rise to a point in $\overline{C}_o$. We make for these points the following definition.

Definition 3.2.

The p'-rational points in $\overline{C}_o$ which correspond to the σ-stable semi-simple conjugacy classes of G will be called the σ-invariant points in $\overline{C}_o$.

The following proposition gives a necessary and sufficient condition for a point x_o to be σ-invariant.

Proposition 3.3.

Let x_o be a p'rational point in $\overline{C}_o$. Then x_o is a σ-invariant point in $\overline{C}_o$ if and only if $x_o \equiv q\omega(x_o)$ mod Y, for some $\omega \in W$.

Proof:

Suppose first that x_o is a σ-invariant point. Let $t_o = h(\chi_{x_o})$ be the element of the torus T_o which represents the σ-stable semi-simple class corresponding to x_o as in 2.4(b'). Thus also the element $\sigma(t_o) = t_o^q = h(\chi_{qx_o})$ is in this class. By 2.4(b) the points x_o and qx_o are in the same W_α-orbit. Suppose that the point x_o lies in the closure of the (ω,y)-alcove $\mathcal{C}_{\omega,y}$, for some $\omega \in W$, $y \in Y$. Then the point qx_o lies in the closed simplex $\omega(\overline{C}_o) + y$.

Thus the point $q\omega^{-1}(x_o)-\omega^{-1}(y)$ is the unique point in $\overline{C}_o$ which determines the W_α-orbit of qx_o. But since x_o lies in $\overline{C}_o$ and is in the same W_α-orbit as qx_o, we must have $x_o = q\omega^{-1}(x_o)-\omega^{-1}(y)$. This gives the result in one direction. Conversely, let x_o be a p'-rational point in $\overline{C}_o$ and let $x_o = q\omega(x_o) + y$, for some $\omega \in W$, $y \in Y$. Then we have $\chi_{x_o} = \chi_{q\omega(x_o)}$ as $\chi_y = 1$. Therefore, $t_o = h(\chi_{x_o}) = h(\chi_{q\omega(x_o)}) = \omega(t_o^q)$. Thus t_o^q is in the W-orbit $\overline{t}_o$ of t_o and so $\overline{t}_o$ is a σ-invariant W-orbit. By 1.3 this means that $[t_o]_G$ is a σ-stable class as required.

Notice that if we work backwards in the second part of the proof we obtain the first. For, as x_o is σ-invariant point by definition the class $[t_o]_G$, $t_o = h(\chi_{x_o})$, is σ-stable and so by 1.3, t_o and $\omega(t_o^q)$ are conjugate for some $\omega \in W$. Thus $\chi_{x_o} = \chi_{q\omega(x_o)}$ which implies $x_o \equiv q\omega(x_o) \bmod Y$. But the above proof of 3.3. points out for what elements $\omega \in W$ and $y \in Y$ we have $x_o = q\omega(x_o)+ y$.

The question now is how the σ-invariant points are distributed in $\overline{C}_o$. To answer this we shall relate the σ-invariant points with the Brauer complex. We will see below that the σ-invariant points have to lie on faces in the Brauer complex which lie on faces of $\overline{C}_o$ of the same type and each face in the Brauer complex contains at most one σ-invariant point. More precisely, this is described by the following theorem which characterizes the σ-invariant points.

Theorem 3.4.

(a) The closure of each ℓ-dimensional face in the Brauer complex B_o contains exactly one σ-invariant point.

(b) Let $\mathfrak{A}_{\omega,y}$ be an alcove (i.e. an ℓ-dimensional face) in B_o, and let F be a face of $\mathfrak{A}_{\omega,y}$ which has the smallest dimension among the faces of $\mathfrak{A}_{\omega,y}$ of which the types are the same as the types of the faces of $\overline{C}_o$ on which they lie. Then F is unique and contains the σ-invariant point lying in $\overline{\mathfrak{A}}_{\omega,y}$. So in particular, the regular σ-invariant points in $\overline{C}_o$ are in ℓ-dimensional faces of B_o.

(c) The closures of two distinct ℓ-dimensional faces in B_o contain distinct σ-invariant points. Thus we have a 1-to-1 correspondence beteeen σ-invariant points in $\overline{C}_o$ and (ω,y)-alcoves in B_o.

Proof:

(a) We suppose that for some $\omega \in W$ and $y \in Y$ we have

$$(*) \quad \frac{1}{q}\, r_o(\omega(x) + y) \le 1, \ \frac{1}{q}\, r_i(\omega(x) + y) \ge 0, \ i = 1,2,\dots,\ell,$$

for all $x \in \overline{C}_o$. In other words the closed simplex $\overline{\mathfrak{A}}_{\omega,y}$ lies in $\overline{C}_o$. Thus because of (*) we also have

$$\overline{\mathfrak{A}}^{k}_{\omega,y} \subset \overline{\mathfrak{A}}^{k-1}_{o}, \quad k \in Z_{\ge 1}.$$

Therefore applying the affine transformation

$$d\left(\frac{1}{q}\, y + \frac{1}{q^2}\, \omega(y) + \dots + \frac{1}{q^{k-1}}\, \omega^{k-2}(y)\right) \omega^{k-1}$$

on the last inclusion we get $S_k \subset S_{k-1}$, where

$$S_k = \overline{\mathfrak{A}}^{k}_{\omega^k,\ \sum_{i=0}^{k-1} q^{k-(i+1)} \omega^i(y)}.$$

Doing this for all $k \in Z_{\geq 1}$, we obtain the infinite chain of simplices $\bar{C}_o = S_o \supset S_1 \supset S_2 \supset \ldots \supset S_k \supset \ldots$. Let n be the order of ω. As $k \to \infty$, the simplex $\bar{\mathfrak{a}}_o^k$ tends to the origin and so the above chain tends to the point

$$\sum_{i=0}^{\infty} \frac{1}{q^{in+1}} y + \sum_{i=0}^{\infty} \frac{1}{q^{in+2}} \omega(y) + \ldots + \sum_{i=0}^{\infty} \frac{1}{q^{in+n}} \omega^{n-1}(y)$$

$$= \frac{1}{q} z + \frac{1}{q^{n+1}} z + \frac{1}{q^{2n+1}} z + \ldots = \sum_{j=0}^{\infty} (\frac{1}{q^n})^j (\frac{1}{q} z)$$

$$= \frac{1/q}{1-1/q^n} z = \frac{q^{n-1}}{q^n-1} z, \text{ where } z = y + \frac{1}{q} \omega(y) + \ldots + \frac{1}{q^{n-1}} \omega^{n-1}(y).$$

Therefore the point $x_o = \frac{q^{n-1}}{q^n-1} z$ lies in $\bar{\mathfrak{a}}_{\omega,y}$ and is a p'-rational point. On the other hand, we have $q\omega^{-1}(x_o)-\omega^{-1}(y) =$ $\frac{q^n}{q^n-1} (\omega^{n-1}(y) + \frac{1}{q} y + \ldots + \frac{1}{q^{n-1}} \omega^{n-2}(y))-\omega^{n-1}(y) = \frac{q^{n-1}}{q^n-1} = x_o$.

Thus by the previous proposition x_o is a σ-invariant point. Moreover, x_o is the unique σ-invariant point in the simplex $\bar{\mathfrak{a}}_{\omega,y}$. For, if x_o' were another point in $\bar{\mathfrak{a}}_{\omega,y}$ satisfying the condition $x_o' = q\omega^{-1}(x_o')-\omega^{-1}(y)$, then we would have $x_o-x_o' = q\omega^{-1}(x_o-x_o')$. This implies $x_o = x_o'$ since the Killing form on H is W-invariant. This gives (a).

(b). We keep the notation as in (a). We saw that the σ-invariant point x_o which is in $\bar{\mathfrak{a}}_{\omega,y}$ lies in every simplex S_k, for all $k \in Z^+$, and has the form $x_o = \frac{q^{n-1}}{q^n-1} \sum_{i=0}^{n-1} \frac{1}{q^i} \omega^i(y)$. We may assume $x_o \neq 0$ and so $x_o \neq \frac{1}{q}y$ (the case $x_o = 0$ is trivial).

Now the bounding hyperplanes of $\overline{\mathfrak{a}}_{\omega,y}$ are of the form $H_{\omega(r_i)}, \frac{1}{q}\,\omega(r_i)(y)$, $i = 1,2,\ldots,\ell$, and $H_{\omega(r_o)}, \frac{1}{q}(1+\omega(r_o)(y))$. Let us suppose that x_o lies on a face F_J of $\overline{C}_o$. We prove that x_o lies also on the face $F^1_{J\ \omega,y}$ of $\mathfrak{a}_{\omega,y}$ of type J. From the definition of the faces, this is equivalent to show that each relation $r_i(x_o) = 0, r_j(x_o) > 0$, $i, j \neq 0$, $r_o(x_o) = 1$, $r_o(x_o) < 1$ separately holds if and only if respectively each relation $\omega(r_i)(x_o) = \frac{1}{q}\omega(r_i)(y)$, $\omega(r_j)(x_o) > \frac{1}{q}\,\omega(r_i)(y)$ $i, j \neq 0$, $\omega(r_o)(x_o) = \frac{1}{q}\,(1 + \omega(r_o)(y))$, $\omega(r_o)(x_o) < \frac{1}{q}\,(1 + \omega(r_o)(y))$ separately holds. So let us prove that

$$r_i(x_o) = 0 \iff \omega(r_i)(x_o) = \frac{1}{q}\,\omega(r_i)(y).$$

We have $r_i(x_o) = 0 \iff \dfrac{q^{n-1}}{q^n-1} \displaystyle\sum_{m=0}^{n-1} \frac{1}{q^m}\, r_i(\omega^m(y)) = 0$

$$\iff \sum_{m=0}^{n-1} \frac{1}{q^{m+1}}\ \omega(r_i)(\omega^{m+1}(y)) = 0$$ (as the Killing form is W-invariant) $\iff \displaystyle\sum_{m=0}^{n-2} \frac{1}{q^{m+1}}\ \omega(r_i)\ (\omega^{m+1}(y)) = -\frac{1}{q^n}\ \omega(r_i)(y)$

(adding in both sides the term $\omega(r_i)(y)$ we get)

$$\iff \sum_{m=0}^{n-1} \frac{1}{q^m}\ \omega(r_i)(\omega^m(y)) = \frac{q^n-1}{q^n}\,\omega(r_i)(y)$$

$$\iff \frac{q^{n-1}}{q^n-1} \sum_{m=0}^{n-1} \frac{1}{q^m}\,\omega(r_i)(\omega^m(y)) = \frac{1}{q}\,\omega(r_i)(y)$$ as required.

The equivalence $r_i(x_o) > 0 \iff \omega(r_j)(x_o) > \frac{1}{q}\,\omega(r_j)(y)$ is proved in the same way. Let us prove the equivalence

$r_o(x_o) < 1 \iff \omega(r_o)(x_o) < \frac{1}{q}\ (1 + \omega(r_o)(y))$.

We have $r_o(x_o) < 1 \iff \sum_{m=0}^{n-1} \frac{1}{q^{m+1}} \omega(r_o)(\omega^{m+1}(y) < \frac{q^n-1}{q^n}$

$$\iff \sum_{m=0}^{n-2} \frac{1}{q^{m+1}} \omega(r_o)(\omega^{m+1}(y)) < \frac{q^n-1-\omega(r_o)(y)}{q^n}$$

(adding the term $\omega(r_o)(y)$ in both sides)

$$\iff \sum_{m=0}^{n-1} \frac{1}{q^m} \omega(r_o)(\omega^m(y)) < \frac{(q^n-1)(1+\omega(r_o)(y))}{q^n}$$

$$\iff \frac{q^{n-1}}{q^n-1} \sum_{m=0}^{n-1} \frac{1}{q^m} \omega(r_o)(\omega^m(y)) < \frac{1}{q}(1+\omega(r_o)(y))$$

as required. Similarly, we obtain the equivalence

$$r_o(x_o) = 1 \iff \omega(r_o)(x_o) = \frac{1}{q}(1+\omega(r_o)(y)).$$

Therefore the σ-invariant point x_o lies on F_J if and only if lies on $F^1_{J,\omega,y}$.

For the rest of the claim in (b) we look at the relative orientation and position of the simplex S_k with respect to S_{k-1}. Suppose that a face $F^1_{J,\omega,y}$ of S_1 lies on the face F_J of $\overline{C}_o$. We apply induction on k to show that the face of S_k of type J lies on F_J, for all $k \in \mathbb{N}$. Since the orientation and the position of S_2 with respect to S_1 is the same as that of S_1 with respect to $\overline{C}_o$, the face of S_2 of type J lies on $F^1_{J,\omega,y}$ and so on F_J. Now suppose that the face of S_k of type J lies on F_J. Then since the relative orientation and position of S_{k+1} with respect to S_k is the same as that of S_k with respect to S_{k-1} we see that the face of S_{k+1} of type J lies on the face of S_{k-1} of type J and so, by the induction hypothesis, on F_J. Taking the limit as

$k \to \infty$ (as we did in (a)) we get a σ-invariant point on the closure of the face F_J. Doing this for all faces of S_1 which lie on faces of $\overline{C}_o$ of the same type we see that the intersection of their closures must be the face of S_1 which contains the σ-invariant point lying in S_1. From the first part of the proof of (b), this face must lie on a face of $\overline{C}_o$ of the same type. This completes the claim in (b).

(c) Since two distinct ℓ-dimensional faces in B_o obviously cannot share a face which lies on a face of $\overline{C}_o$ of the same type, (c) is a consequence of (a) and (b).

Corollary 3.4.(a)

(a) Each σ-invariant point in $\overline{C}_o$ is of the form

$\frac{q^{n-1}}{q^n-1}(y + \frac{1}{q}\,\omega(y) + .. + \frac{1}{q^{n-1}}\,\omega^{n-1}(y))$, for some $w \in W$, $y \in Y$, where n is the order of ω.

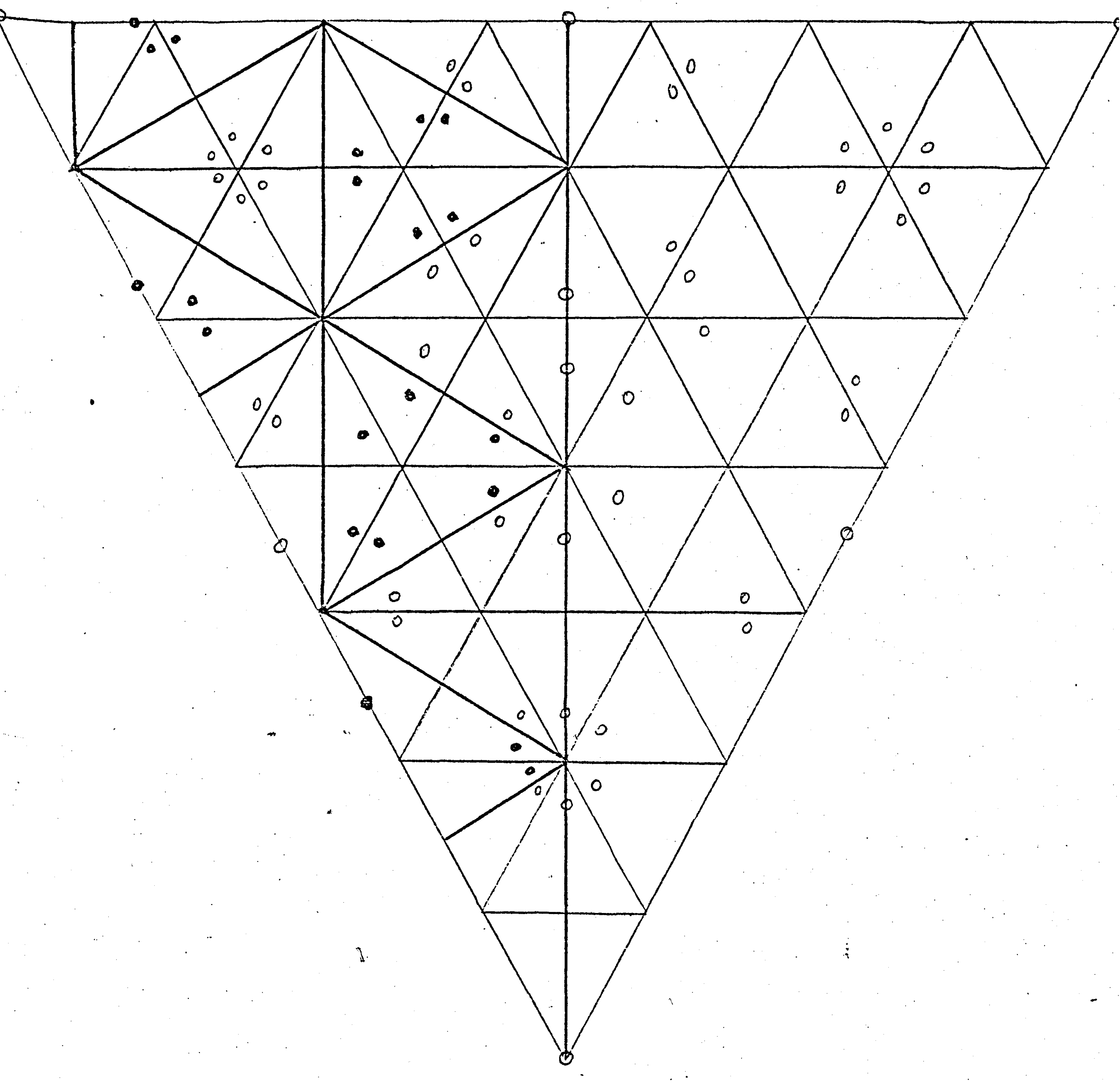
(b) The number of σ-stable semi-simple conjugacy classes of G is q^ℓ.

Proof:

(a) is in the proof of the theorem.

(b) is a well known result (see [14]) which also is an immediate consequence of the above theorem since the number of ℓ-dimensional faces of B_o is q^ℓ.

The figure below is an example for q=7 showing how the σ-invariant points are distributed on the faces of the Brauer complex B_o for types A_2 and G_2 which have been drawn in the same figure. The blank points are the σ-invariant points for A_2. The bold points together with the blank points in G_2 consist of the σ-invariant points for G_2.



In the next chapter we shall derive some further properties of the σ-invariant points in relation with the Brauer complex which will be concerned with the occurence of the possible centralizers of semi-simple elements of G_σ.

CHAPTER 4

CENTRALIZERS OF SEMI-SIMPLE ELEMENTS IN THE FINITE CHEVALLEY GROUPS

We have seen in Chapter 2 that the p'-rational points in $\overline{C}_o$ can be used to give information about the structure of G. So it is natural one to expect that the σ-invariant points, and so the Brauer complex, can be used to play an analogous rôle for the finite groups G_σ. Notice that we have already seen that we were able to deduce from the Brauer complex the well known result for the number of the semi-simple conjugacy classes in $G_{sc,\sigma}$.

The recent Carter's work together with the Theorem 2.11 allow us to define certain pairs $(E_J,[\omega])$ which will determine the structure of the possible centralizers of semi-simple elements in G_σ, where E_J is the type of some proper subset Δ_J of the extended Dynkin diagram of G and $[\omega]$ is a conjugacy class in the normalizer Ω_J of Δ_J in W.

Deligne and Lusztig, in their fundamental work [9], have constructed certain families of irreducible complex representations of the group $(G^*_{sc})_\sigma$ where G^*_{sc} is the dual group (see [9]) of G_{sc}, that is an adjoint group whose root system is isomorphic with the dual root system of G_{sc}. In [9], these families correspond one-to-one to the classes of an equivalence relation on the semi-simple elements of $G_{sc,\sigma}$ defined as follows: we say that two semi-simple elements x, y in $G_{sc,\sigma}$ are equivalent if their centralizers $C_G(x)$, $C_G(y)$ are G_σ-conjugate. Because of their nature, these representations have been called semi-simple irreducible representations

The pairs $(E_J,[\omega])$ can be used to parametrize these families and give the degrees of the semi-simple irreducible representations of $(G^*_{sc})_\sigma$. For, the irreducible representations of $(G^*_{sc})_\sigma$ belonging in a certain family have the same degree given by the p'part of

$\frac{|G_{sc,\sigma}|}{|C_{G_{sc,\sigma}}(x)|}$, where x is a semi-simple element which defines the class corresponding to the family.

Finally, the geometry of the Brauer complex will tell us for which prime powers q these families occur. This also can be done by algebraic methods described in [6].

4.1. Review of Carter's work

Let $G = G_{\Pi}(K)$ be a simple Chevalley group defined as in §1.1, and σ the Frobenius endomorphism of G. In a recent paper [6], R. Carter has determined the structure of the "connected reductive" subgroups of maximal rank in G_{σ}. These subgroups are the σ-stable elements of the σ-stable connected reductive subgroups of maximal rank in G. In particular, this work applies to the connected centralizers of semi-simple elements in G_{σ}.

In the present paragraph we present a brief summary of this work.

Let T be a σ-stable maximal torus of G and Φ the root system of G with respect to T. For a closed subroot system Φ_1 of Φ (i.e. $r, s \in \Phi_1$, $r + s \in \Phi$ imply $r + s \in \Phi_1$) there corresponds a σ-stable connected reductive subgroup G_1 of G generated by T along with all root subgroups X_r, $r \in \Phi_1$. Conversely, any σ-stable connected reductive subgroup G_1 of maximal rank in G has this property; that is, G_1 possesses a σ-stable maximal torus T of G and is generated by T along with root subgroups of G which are defined by roots running over a closed subroot system Φ_1 of $\Phi(=\Phi(T))$, the root system of G_1.

We assume G_1 is such a subgroup and consider a fundamental basis Δ_1 of its root system Φ_1. We denote Ω_1 the normalizer of Δ_1 in W.

Let $\mathcal{C}_1$ be the set of all σ-stable conjugates of G_1 in G. If G_1^g is one of them, then we may assume that the maximal torus T^g of G_1^g is σ-stable (since otherwise as G_1^g contains σ-stable maximal tori of G and as all the maximal tori in G_1^g are conjugate in G_1^g we can replace T^g by T^{gx} for some $x \in G_1^g$ to obtain a σ-stable torus in G_1^g of the form we need). Thus we have $\sigma(G_1^g) = G_1^g$ and $\sigma(T^g) = T^g$, which shows that $g^{-1}\sigma(g) \in N_G(G_1) \cap N_G(T)$.

We have seen (Proposition 1.1) that there is a bijection between the G_σ-orbits of σ-stable maximal tori in G and the conjugacy classes in W. Now this has been generalized in [6] as follows.

<u>Proposition 4.1</u> ([6])

We make G_σ to act on $\mathcal{C}_1$ by conjugation. Then each conjugacy class [ω] of Ω_1 gives rise to the orbit in $\mathcal{C}_1 \backslash G_\sigma$ represented by the group G_1^g, where $\pi(g^{-1}\sigma(g)) = \omega$, π being the natural homomorphism of $N_G(T)$ onto W. The map $[\omega]_{\Omega_1} \longrightarrow \overline{G}_1^g$ is a bijection, where $\overline{G}_1^g$ denotes the orbit of G_1^g in $\mathcal{C}_1 \backslash G_\sigma$.

When G_1 is a torus, then $\Omega_1 = W$, and Proposition 1.1 is an immediate consequence of 4.1.

Now the groups in a certain orbit in $\mathcal{C}_1 \backslash G_\sigma$ have the same structure as they are conjugate in G_σ . In particular they have the same order.

In [6] the structure of $(G_1^g)_\sigma$ is described if we know the action on Δ_1 of a representative element in the corresponding class in Ω_1. This is given as follows.

Let $M = \langle X_r;\ r \in \Phi_1 \rangle$, where X_r is the root subgroup attached to the root r. M is the derived subgroup of G_1 which is semi-simple. Let $S = Z(G_1)^o$ be the identity component of the centre of G_1. Then $M \cap S = A$ is finite and we have $G_1 = MS$. Now the groups M, S are σ-stable, being characteristic subgroups of G_1, and G_1 is $\mathbb{F}_q$-isogenous to the direct product $M \times S$. For, both the connected groups G_1 and $M \times S$ are $\mathbb{F}_q$-isogenous to $M/A \times S/A$ which is isomorphic to $M \times S/_{A \times A}$. The following proposition shows that we have $|(G_1)_\sigma| = |M_\sigma|\ |S_\sigma|$.

Proposition 4.2. ([1]).

If k is a subfield of K and H_1, H_2 are two connected algebraic groups defined over k which are also k-isogenous, then the groups of the k-rational points of H_1 and H_2 respectively have the same number of elements.

The same applies to the σ-stable G-conjugates of G_1. That is, if G_1^g, $g \in G$, is σ-stable, then $|(G_1^g)_\sigma| = |(M^g)_\sigma|\ |(S^g)_\sigma|$. This will need later to compute the orders of the possible connected centralizers of semi-simple elements of G_σ.

We denote by P_1 the Z-lattice of Δ_1 and by $\overline{P}_1$ the subgroup of the lattice X(T) which is defined such that $\overline{P}_{1/P_1}$ is the torsion part of $X(T)/p_1$. In other words $\overline{P}_1$ is the subgroup of X(T) consisting of all the weights which are linear combinations of roots in Δ_1 over the rational numbers.

With the above notation we have the following proposition.

Proposition 4.3. ([6])

Let G_1^g be a representative of an orbit in $\mathcal{C}_1\backslash G_\sigma$ such that $\pi(g^{-1}\sigma(g)) = \omega \in \Omega_1$.

Then the following hold.

(a) $(M^g)_\sigma \simeq M_{\sigma\tau}$, where τ denotes the graph automorphism of M determined by the symmetry of Δ_1 induced by ω . (see Chapter 12, [4]).

(b) $(S^g)_\sigma \simeq S_{\sigma,\omega} \simeq X(T)/\overline{P}_1 \Big/ (q\omega-1)\, X(T)/\overline{P}_1$

The above results can be applied to the σ-stable connected centralizers of semi-simple elements of G. This we shall discuss in the next paragraph along with the question when a σ-stable connected reductive subgroup is the centralizer of a semi-simple element of G_σ.

4.2. Application of the Brauer complex to the structure of G_σ.

We have seen in §2.2. that a connected reductive subgroup of maximal rank in G is the connected centralizer of some semi-simple element of G if and only if its root system is W-conjugate to some Φ_J, $J \subsetneq I_o$, provided p is not a bad prime. Here we are interested for such σ-stable centralizers. So let G_J be such a subgroup with root system conjugate to some Φ_J, $J \subsetneq I_o$. We may suppose that G_J contains the maximal torus T_o (notation as in 1.2). For, otherwise we take a conjugate of G_J containing T_o. Let $\mathcal{C}_J$ be the set of all σ-stable conjugates of G_J as in 4.1. So in the first place for each proper subset J of I_o, we have the set $\mathcal{C}_J\backslash G_\sigma$ of G_σ– orbits in $\mathcal{C}_J$, which is characterized by the type E_J of Φ_J. In the second place, by 4.1, each orbit in $\mathcal{C}_J\backslash G_\sigma$ is characterized by a conjugacy class $[\omega]$ of Ω_J. Thus the G_σ -orbits of the set of all σ-stable connected centralizers of semi-simple elements of G are parametrized by pairs $(E_J,[\omega])$ which we shall call *families*. Now we can ask: given a family $(E_J,[\omega])$ is there any centralizer in the corresponding G_σ–orbit which is the centralizer for a semi-simple element in G_σ. The answer to this is given by the following theorem due to R. W. Carter.

Theorem 4.4. [6]

Let $(E_J,[\omega])$ be a family as above, where $\omega = \pi(g^{-1}\sigma(g))$. Then G_J^g is the centralizer of some semi-simple element in G_σ if and only if there is a complex character of the group $\Gamma_{J,\omega} = X(T_o)/P_J\big/(q\omega-1)\ X(T_o)/P_J$, which does not annihilate any root in $\overline{\Phi}_J - \Phi_J$ provided q is sufficiently large, where $\overline{\Phi}_J = \Phi \cap \mathbb{Q}\Phi_J$ and $P_J = \mathbb{Z}\Phi_J$.

When G_J^g in 4.4. occurs to be the centralizer of a semi-simple element of G for a given q, then we shall say that the family $(E_J,[\omega])$ occurs for this q.

Now if we know the structure of each finite abelian group $\Gamma_{J,\omega}$, $J \subsetneq I_o$, $\omega \in \Omega_J$, then from Theorem 4.4, one can obtain the conditions on q for which the corresponding families $(E_J,[\omega])$ occur.

In the special case when G is the simply connected group G_{sc} we can use the Brauer complex instead of the Theorem 4.4, in order to find the conditions on q, for the occurrence of the families $(E_J,[\omega])$. This we shall do next assuming that $G = G_{sc}$.

The reason which enables us to use the Brauer complex for the present needs becomes clear from the Corollary which follows the following lemma.

Lemma 4.5.

Let $\mathcal{A}_{\omega,y}$ be an (ω,y)-alcove in $\overline{C}_o$ and let x_o be the σ-invariant point in $\mathcal{A}_{\omega,y}$. Then $\omega \in \Omega_J$, where J is the type of the face of C_o on which x_o lies.

Proof:

For $J = \emptyset$, we have $\Omega_J = W$ and so the lemma is true. Let us suppose that $J \neq \emptyset$. We consider the hyperplanes on which the face $F^1_{J,\omega,y}$ of the alcove $\mathcal{A}_{\omega,y}$ lies. These are the hyperplanes

$$H_{j,\omega,y} = H_{\omega(r_j),\ \frac{1}{q}\omega(r_j)(y)}, \text{ for } 0 \neq j \in J \text{ and}$$

$$H_{o,\omega,y} = H_{\omega(-r_o),\ -\frac{1}{q}(1-\omega(-r_o)(y))}, \text{ if } 0 \in J.$$

Now since x_o is a σ-invariant point lying on F_J we have seen (Theorem 3.4) that the face $F^1_{J,\omega,y}$ of $\mathcal{A}_{\omega,y}$ has to lie on the face F_J of C_o, and also x_o does not lie in the closure of any other ℓ-dimensional face of B_o apart from $\mathcal{A}_{\omega,y}$.

Therefore if $F^1_{J_1,\omega,y}$ is a face of $\mathfrak{A}_{\omega,y}$ of which the closure contains the face $F^1_{J,\omega,y}$, then $F^1_{J_1,\omega,y}$ will lie on a face F_{J_2} of C_o of which the closure contains the face F_J and $\dim F_{J_2} = \dim F^1_{J_1,\omega,y}$. In particular, this holds for the faces of $\mathfrak{A}_{\omega,y}$ of dimension $\ell-1$ and of which the closures contain the face $F^1_{J,\omega,y}$. But by definition, the face F_J (resp. the face $F^1_{J,\omega,y}$) lies on the intersection of the hyperplanes H_j, $0 \neq j \in J$ and H_o if $0 \in J$ (resp. $H_{j,\omega,y}$, $j \in J$). Thus the two sets $\{H_j;\ 0 \neq j \in J\}$ and $\{H_{j,\omega,y};\ j \in J\}$ of hyperplanes must coincide, where we have to add in the first set the hyperplane H_o, if $0 \in J$. This gives that the element ω normalizes Δ_J i.e. $\omega \in \Omega_J$ as required.

Corollary 4.6.

A family $(E_J,[\omega])$ occurs for a given q if and only if there exists some (ω',y)-alcove $\mathfrak{A}_{\omega',y}$ in $\overline{C}_o$ such that $\omega' \in [\omega]$ and the σ-invariant point lying in $\mathfrak{A}_{\omega',y}$ is on the face of C_o of type J.

Proof:

Straightforward by the definition of a family, Proposition 3.3. and Lemma 4.5.

Now we look what conditions must satisfy the point y for an (ω,y)-alcove in the Brauer complex B_o.

In the proof of the Lemma 4.5. we have seen that if the σ-invariant point x_o in $\mathfrak{A}_{\omega,y}$ lies on F_J, then the hyperplanes H_j, $j \in J$ which define F_J coincide with the hyperplanes $H_{j,\omega,y}$, $j \in J$. This forces y to satisfy one of the following set of inequalities.

(a) $r_o(y) \leq q-1$, $r_j(y) = 0$, $0 \neq j \in J$, $r_i(y) > 0$, $i \notin J$ when either $0 \notin J$ or if $0 \in J$, $\omega(r_o) = r_o$.

(b) $r_o(y) = q$, $r_k(y) = 1$, $r_j(y) = 0$, $0, k \neq j \in J$, $r_i(y) > 0$, $i \notin J$. when $\omega(-r_o) = r_k$.

In particular, if $0 \in J$, then either $r_o(y) = q-1$ or $y(r_o) = q$. Conversely, if $\mathfrak{A}_{\omega,y} \subset \overline{C}_o$ and $\omega \in \Omega_J$ and y satisfies one of the above set of inequalities, then the σ-invariant point in $\overline{\mathfrak{A}}_{\omega,y}$ must lie on the face $F^1_{J,\omega,y}$ and so on F_J. This is because $\overline{\mathfrak{A}}_{\omega,y}$ contains only one σ-invariant point and this point by 4.5. must lie on F_J.

Thus given a conjugacy class $[\omega]$ in Ω_J, to see if the family $(E_J,[\omega])$ occurs for a given q we have to look if there exists some element ω' in $[\omega]$ and $y \in Y$ such that $\mathfrak{A}_{\omega,y} \subset \overline{C}_o$ and y to satisfy one of the above set of inequalities. More precisely, if we write $y = \sum_{\rho=1}^{\ell} \mu_\rho \gamma_\rho$, then (a) and (b) say respectively that

(a)' $\mu_j = 0$, $0 \neq j \in J$, $\mu_i > 0$, $i \notin J$, $\sum_{i \in I-J} \mu_i n_i \leq q-1$

when either $0 \notin J$ or if $0 \in J$, $\omega(r_o) = r_o$.

(b)' $\mu_k = 1$, $\mu_j = 0$, $0, k \neq j \in J$, $\mu_i > 0$, $i \notin J$, $n_k + \sum_{i \in I-J} \mu_i = q$.

when $\omega(-r_o) = r_k$,

where $I = I_o - \{0\}$ and the n_i's are the coefficients of the highest root r_o.

Since $y \in Y$, we must have $y = \sum_1^{\ell} z_\nu h_\nu$, where

$$(*)\quad z_\nu = \frac{1}{d} \sum_{i \in I-J} (-1)^{i+\nu} \mu_i C_{i,\nu} = \frac{1}{d} \sum_{i \in I-J} \mu_i |C_{i,\nu}|.$$

as we have $(-1)^{i+j} C_{ij} = |C_{ij}|$ for the determinants C_{ij} of the $(\ell-1) \times (\ell-1)$ matrices obtained from the transpose matrix of the Cartan matrix A by deleting the i^{th} row and j^{th} column. Here d is the determinant of A. Thus we have to look for the existence of some sŏlution of one of the above sets of inequalities which satisfy (*). This will give the condition on q for the occurrence of the family $(E_J,[\omega])$.

We shall see in Chapter 5 that given a Dynkin diagram of type E_J for some $J \subsetneq I_o$, then except for the bad primes and sufficient large q, the face F_J contains a σ-invariant point. Although we do not have a direct proof for this (but only this results from case by case) we shall give below direct proofs for particular types of faces.

We make the following definitions.

<u>Definition 4.7.</u>

The subsets of the Dynkin diagram of Φ will be called strictly parabolic subsets of the extended Dynkin diagram $\tilde{\Delta}$. A subset Δ_J of $\tilde{\Delta}$ which is W-conjugate to a strictly parabolic subset will be called parabolic subset, otherwise we shall call Δ_J a non-parabolic subset of $\tilde{\Delta}$. With this terminology we have the following Corollary of the Theorem 4.4.

<u>Corollary 4.8</u>.

If Δ_J is a parabolic subset of $\tilde{\Delta}$, then the families $(E_J,[\omega])$, $\omega \in \Omega_J$, occur for all q sufficiently large.

<u>Proof</u>:

Since two connected reductive subgroups of G of maximal rank which have W-conjugate subroot systems are G-conjugate, we may assume that Δ_J is a strictly parabolic subset. In this case we have $\overline{\Phi}_J = \Phi_J$ and the Theorem 4.4 applies to give the required result.

Corollary 4.9.

Let F_J be a face of C_o such that Δ_J is a strictly parabolic type. Then, for sufficiently large q, there exists a point y in $F_J \cap Y$. Moreover, the elements $\omega \in W$ for which the alcoves $\mathfrak{A}_{\omega,y}$ lie in $\overline{C}_o$ and have their faces of type J on F_J give all the conjugacy classes of Ω_J.

This follows from 4.6 and 4.8.

The existence of a point in $F_J \cap Y$ when q is sufficiently large, is equivalent with the existence of a solution of the inequality $\sum_{i \in I-J} \mu_i n_i < q-1$ in (a)' which satisfy (*). Observe that if we take the smallest values of μ_i's, $i \in I-J$, such that the z_ν's in (*) to be integers then the inequality $\sum_{i \in I-J} \mu_i n_i < q-1$ is satisfied provivded q is sufficiently large.

We make now the following conjecture.

Conjecture

Let $F^1_{J,\omega,y}$ be a face of an alcove $\mathfrak{A}_{\omega,y}$ lying on a face $F_{J'}$ of C_o which is not of strictly parabolic type and $\dim F^1_{J,\omega,y} = \dim F_{J'}$. We suppose that there exists a face $F^1_{J_1,\omega,y}$ of $\mathfrak{A}_{\omega,y}$ with the following two properties.

(1) The closure of $F^1_{J_1,\omega,y}$ contains $F^1_{J,\omega,y}$

(2) $F^1_{J_1,\omega,y}$ lies on a face of C_o of dimension greater than the dimension of its own.

Then the vertex $\frac{1}{q}y$ of $\mathfrak{A}_{\omega,y}$ lies on the face $F_{J'}$, and this happens when p is a bad prime number.

If this conjecture is true we can deduce the following Corollary.

<u>Corollary 4.10</u>.

If Δ_J is a subset of $\tilde{\Delta}$ which is not W-conjugate to any other subset of $\tilde{\Delta}$, then except for the bad primes and sufficiently large q, the face F_J contains a σ-invariant point.

<u>Proof</u>:

It can happen (for groups of type E_7 and E_8) that although the subset Δ_J is not W-conjugate to any other subset of $\tilde{\Delta}$, to exist another face $F_{J'}$ such that the corresponding subset $\Delta_{J'}$ has Dynkin diagram of the same type as this of Δ_J. So let us first suppose that no other subset of $\tilde{\Delta}$ has the same Dynkin diagram as this of Δ_J. We suppose that Δ_J is not a strictly parabolic subset and that p is not a bad prime (the case of strictly parabolic subsets has been considered in 4.8). Thus the face F_J is on the boundary of the face $F_{\{0\}}$. Let $\bar{\mathfrak{A}}_{\omega,y}$ be an alcove which intersects the face F_J in a face of the same dimension as this of F_J. By our conjecture the vertex $\frac{1}{q}y$ of $\mathfrak{A}_{\omega,y}$ is not on the face F_J and so this face of $\mathfrak{A}_{\omega,y}$, in question, must not be of strictly parabolic type. Let us suppose that F_J is a vertex ν_J; then for all q, the vertex of $\mathfrak{A}_{\omega,y}$ which lies on ν_J must be of type J, since there is no other vertex of C_o with the same Dynkin diagram as this of the vertex ν_J. Thus for the vertices, the Corollary is true for all q (except the bad primes). Now suppose that F_J is not a vertex. For the same reason as in the case of a vertex, the face of $\mathfrak{A}_{\omega,y}$ which lies on F_J and has the same dimension as this of F_J must be the face $F^1_{J,\omega,y}$. If we take q to be sufficiently large, then we can find some alcove $\mathfrak{A}_{\omega,y}$ such that $\bar{\mathfrak{A}}_{\omega,y} \cap F_J$ is the closure of the face $F^1_{J,\omega,y}$ of $\mathfrak{A}_{\omega,y}$. So, for sufficiently large q and except for the bad primes we can find an alcove $\bar{\mathfrak{A}}_{\omega,y}$ intersecting the face F_J in the closure of its face $F^1_{J,\omega,y}$. This means that $F^1_{J,\omega,y}$ is the face of smallest dimension among the faces of $\mathfrak{A}_{\omega,y}$ which lie on faces of C_o of the same type. Thus by Theorem 3.4. the σ-invariant point contained in $\bar{\mathfrak{A}}_{\omega,y}$ lies on F_J.

Now suppose that there is some $\Delta_{J'}$ with the same Dynkin diagram as this of Δ_J. Then the alcoves whose closures intersect the face F_J must intersect F_J in faces of type either J' or J. Suppose that some alcove $\bar{\mathfrak{A}}_{\omega,y}$ intersects F_J in the face $F^1_{J',\omega,y}$. But then, the hyperplanes $H_{j',\omega,y}$, $j' \in J'$ (notation as in 4.5) on which $F^1_{J',\omega,y}$ lies must coincide with the hyperplanes on which F_J lies. This gives $\omega(\Delta_J) = \Delta_{J'}$, which is a contradiction to our assumption.

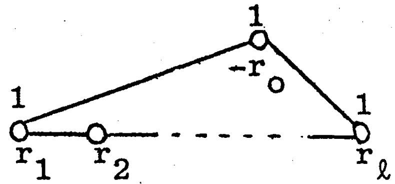
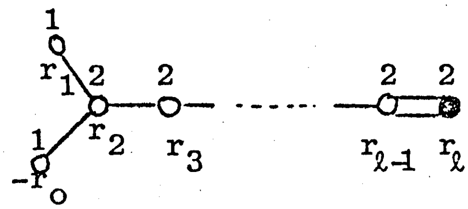
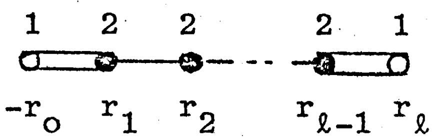
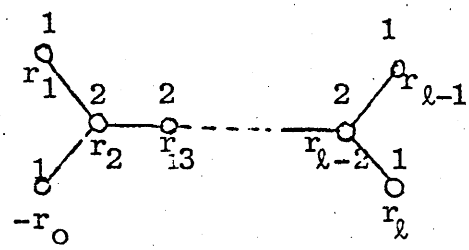
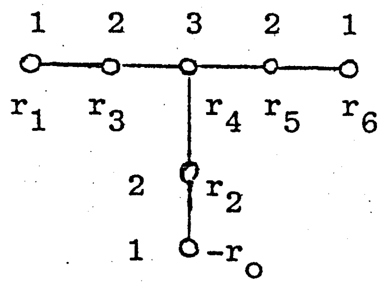
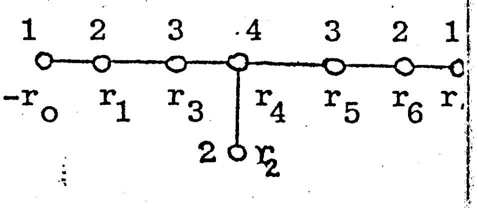
CHAPTER 5

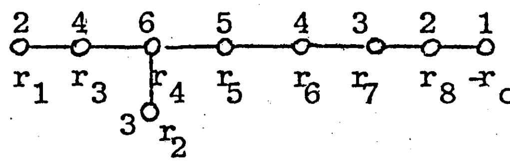
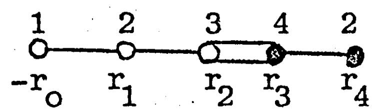
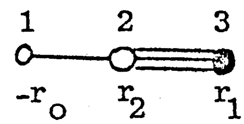
The orders of the connected centralizers of semi-simple elements in the finite exceptional Chevalley groups

In the present chapter we apply the theory developed so far to write down in tables the orders of the connected centralizers of semi-simple elements in the group G_σ of the fixed points of the Frobenius map σ of an exceptional Chevalley group G. That is, G is a group in one of the isogeny classes of type E_6, E_7, E_8, F_4 and G_2. The case when G is a classical group has been treated recently in [7]. Also, we determine the conditions which have to be imposed on q for the occurrence of each family $(E_J,[\omega])$ discussed in Chapter 4.

To obtain these tables it is useful to have detailed information about the indecomposable reduced root systems Φ, each of which can be embedded in a suitable real vector space $\mathbb{R}^n$ with a canonical basis $\{\varepsilon_i\}_{1\le i\le n}$. This information is given in Table 1.

TABLE 1

Type	Chosen basis Δ	Positive roots	Extended Dynkin diagram
A_ℓ	$r_i=\varepsilon_i-\varepsilon_{i+1}$, $1\le i\le \ell$	$\varepsilon_i-\varepsilon_j$, $1\le i<j\le \ell+1$	
B_ℓ	$r_i=\varepsilon_i-\varepsilon_{i+1}$, $r_\ell=\varepsilon_\ell$, $1\le i\le \ell-1$	$\varepsilon_i, \varepsilon_i+\varepsilon_j$, $1\le i,j\le \ell, i\neq j$; $\varepsilon_i-\varepsilon_j$, $1\le i<j\le \ell$	
C_ℓ	$r_i=\varepsilon_i-\varepsilon_{i+1}$, $r_\ell=2\varepsilon_\ell$, $1\le i\le \ell-1$	$2\varepsilon_i, \varepsilon_i+\varepsilon_j$, $1\le i,j\le \ell, i\neq j$; $\varepsilon_i-\varepsilon_j$, $1\le i<j\le \ell$	
D_ℓ	$r_i=\varepsilon_i-\varepsilon_{i+1}$, $r_\ell=\varepsilon_{\ell-1}+\varepsilon_\ell$, $1\le i\le \ell-1$, $\ell\ge 4$	$\varepsilon_i+\varepsilon_j$, $1\le i,j\le \ell, i\neq j$; $\varepsilon_i-\varepsilon_j$, $1\le i<j\le \ell$	
E_6	$r_1=\frac{1}{2}(\varepsilon_1+\varepsilon_8-\sum_1^7 \varepsilon_i)$; $r_2=\varepsilon_1+\varepsilon_2$; $r_i=\varepsilon_{i-1}-\varepsilon_{i-2}$, $3\le i\le 6$	$\pm\varepsilon_i+\varepsilon_j, 1\le i<j\le 5$; $\frac{1}{2}(\varepsilon_8-\varepsilon_7-\varepsilon_6+\sum_1^5(-1)^{\nu_i}\varepsilon_i)$; $\nu_i=0,1$, $\sum_1^5 \nu_i$ even	
E_7	$r_1=\frac{1}{2}(\varepsilon_1+\varepsilon_j-\sum_1^7 \varepsilon_i)$; $r_2=\varepsilon_1+\varepsilon_2$; $r_i=\varepsilon_{i-1}-\varepsilon_{i-2}$, $3\le i\le 7$	$\pm\varepsilon_i+\varepsilon_j, 1\le i<j\le 6$; $\frac{1}{2}(\varepsilon_8-\varepsilon_7+\sum_1^6(-1)^{\nu_i}\varepsilon_i)$; $r_i=0,1$, $\sum_1^6 \nu_i$ odd	

Type	Chosen basis Δ	Positive roots	Extended Dynkin diagram
E_8	$r_1=\frac{1}{2}(\varepsilon_1+\varepsilon_8-\sum_1^7\varepsilon_i)$ $r_2=\varepsilon_1+\varepsilon_2$ $r_i=\varepsilon_{i-1}-\varepsilon_{i-2}, 3\le i\le 8$	$\pm\varepsilon_i+\varepsilon_j, 1\le i<j\le 7$ $\frac{1}{2}\sum_1^8(-1)^{\nu_i}\varepsilon_i$, $\nu_i=0,1,\ \sum_1^8\nu_i$ even	
F_4	$r_1=\varepsilon_2-\varepsilon_3, r_2=\varepsilon_3-\varepsilon_4$ $r_3=\varepsilon_4, r_4=\frac{1}{2}(\varepsilon_1-\varepsilon_2-\varepsilon_3-\varepsilon_4)$	$\varepsilon_i\pm\varepsilon_j, \varepsilon_i, 1\le i<j\le 4$ $\frac{1}{2}(\varepsilon_1\pm\varepsilon_2\pm\varepsilon_3\pm\varepsilon_4)$	
G_2	$r_1=\varepsilon_1-\varepsilon_2$ $r_2=-2\varepsilon_1+\varepsilon_2+\varepsilon_3$	$\varepsilon_1-\varepsilon_2, -\varepsilon_2+\varepsilon_3, -\varepsilon_1+\varepsilon_3$ $-2\varepsilon_1+\varepsilon_2+\varepsilon_3, -2\varepsilon_2+\varepsilon_1+\varepsilon_3$ $2\varepsilon_3-\varepsilon_1-\varepsilon_2$	

In the above table we have shown in black the nodes of the diagrams corresponding to short roots and the numbers standing on the nodes are the coefficients of the highest root with respect to Δ.

For the imposed conditions on q for the occurrence of the families we have seen in Chapter 4 that we need to know the inverse of the transpose of the Cartan matrix of Φ. For the exceptional groups, with which we are dealing here, these matrices are as follows:

E_6:

$$\frac{1}{3}\begin{bmatrix} 4 & 3 & 5 & 6 & 4 & 2 \\ 3 & 6 & 6 & 9 & 6 & 3 \\ 5 & 6 & 10 & 12 & 8 & 4 \\ 6 & 9 & 12 & 18 & 12 & 6 \\ 4 & 6 & 8 & 12 & 10 & 5 \\ 2 & 3 & 4 & 6 & 5 & 4 \end{bmatrix}$$

E_7:

$$\frac{1}{2}\begin{bmatrix} 4 & 4 & 6 & 8 & 6 & 4 & 2 \\ 4 & 7 & 8 & 12 & 9 & 6 & 3 \\ 6 & 8 & 12 & 16 & 12 & 8 & 4 \\ 8 & 12 & 16 & 24 & 18 & 12 & 6 \\ 6 & 9 & 12 & 18 & 15 & 10 & 5 \\ 4 & 6 & 8 & 12 & 10 & 8 & 4 \\ 2 & 3 & 4 & 6 & 5 & 4 & 3 \end{bmatrix}$$

E_8:

$$\begin{bmatrix} 4 & 5 & 7 & 10 & 8 & 6 & 4 & 2 \\ 5 & 8 & 10 & 15 & 12 & 9 & 6 & 3 \\ 7 & 10 & 14 & 20 & 16 & 12 & 8 & 4 \\ 10 & 15 & 20 & 30 & 24 & 18 & 12 & 6 \\ 8 & 12 & 16 & 24 & 20 & 15 & 10 & 5 \\ 6 & 9 & 12 & 18 & 15 & 12 & 8 & 4 \\ 4 & 6 & 8 & 12 & 10 & 8 & 6 & 3 \\ 2 & 3 & 4 & 6 & 5 & 4 & 3 & 2 \end{bmatrix}$$

F_4:

$$\begin{bmatrix} 2 & 3 & 2 & 1 \\ 3 & 6 & 4 & 2 \\ 4 & 8 & 6 & 3 \\ 2 & 4 & 3 & 2 \end{bmatrix}$$

G_2:

$$\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

From these matrices we can deduce that an exceptional Chevalley group $G_\Pi(K)$ is either G_{sc} or G_{ad} and these groups are distinct only in the cases of type E_6 and E_7.

The orders of the possible centralizers determined by a given family $(E_J,[\omega])$ are the same (Proposition 4.2) for every group G in a given isogeny class. To find these orders we apply Proposition 4.3 and Theorem 12.4 [14].

To determine the conjugacy classes in the group Ω_J, for each $J \subsetneq I_0$, we make great use of the material of Carter's paper [5]. For example, for all the types of groups, the orders of the centralizers determined by the families $(\phi,[\omega])$ can be obtained straight from the tables given in [5]. For, on the one hand these centralizers are the maximal tori in G_σ determined from the tori which are determined by twisting the maximal torus T_0 by an element in $[\omega]$. On the other hand, in this case we have $\Omega_\phi = W$ and the conjugacy classes of W correspond to certain diagrams - called admissible diagrams. Such a diagram which corresponds to $[\omega]$, gives the characteristic polynomial $f_\omega(t)$ of ω on X. These polynomials are listed also in [5]. Now the value of $f_\omega(t)$ at q is just the order of the maximal torus $T_\omega(q)$ of G_σ determined from the torus which is obtained by twisting T_0 by ω.

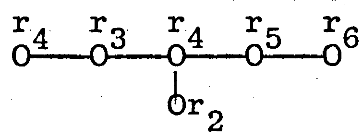
We shall not include the families $(\phi,[\omega])$ in our tables since the reader can have a complete list of these from [5] as has been pointed out above.

Our notation here for the finite Chevalley groups G_σ and their twisted versions, which we met in §4.2, will be the same as in [4] where the orders of these groups are also given.

Let us give now two examples in order to see how the tables which follow have been worked out.

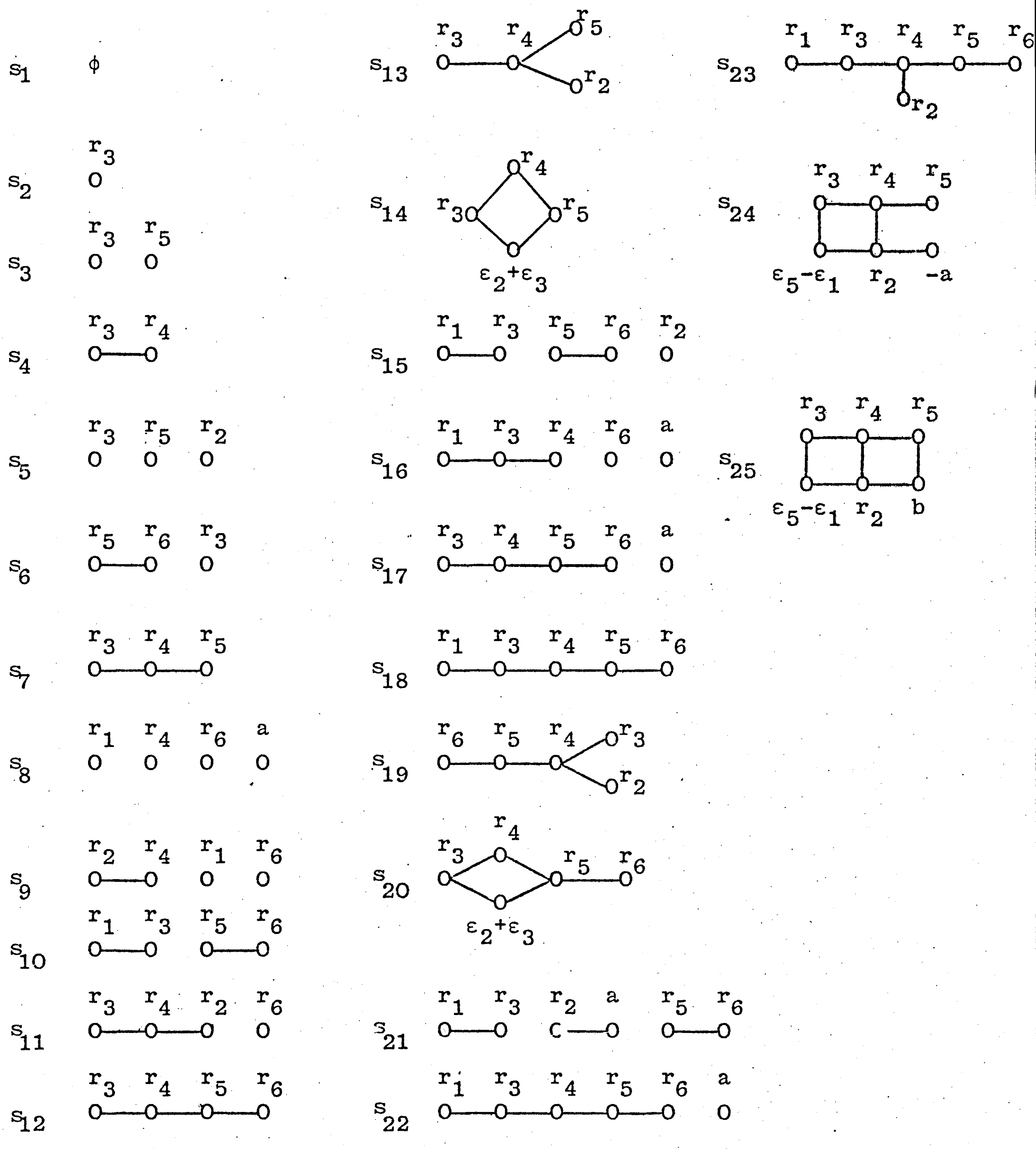
Example 1: We consider the Dynkin diagram of type A_2 obtained from a subset of the extended Dynkin diagram of type E_8. As a subset of type A_2 we choose the subset Δ_J: $\overset{r_8}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{-r_0}{\circ}$. Obviously, this is a parabolic subset and so by Corollary 4.8 all the families $(A_2,[\omega])$, $\omega \in N_W(\Delta_J) = \Omega_J$ do occur for all q sufficiently large. To find the orders of the centralizers it is enough to know a representative element for each conjugacy class $[\omega]$ in Ω_J. We look at the structure of Ω_J.

The Weyl subgroup $W_J^{\perp}$ of the orthogonal subroot system $\Phi_J^{\perp}$ to Φ_J is of type E_6 generated by the reflections in the hyperplanes orthogonal to the roots corresponding to the nodes of the diagram $\Delta_J^{\perp}$:



. We have seen (Proposition 2.2) that $W_J^{\perp}$ is a normal subgroup of Ω_J and $|\Omega_J:W_J^{\perp}| \leq 2$, since Δ_J can only have one non-trivial symmetry. We consider the element $\sigma = \omega_o'\omega_o$, where ω_o,ω_o' are respectively the elements of maximal length in the Weyl subgroups $W(E_7)$ and $W(E_6)$ generated by the roots of the corresponding bases Δ in Table 1. We can easily check now (see p.p. 15) that the element $\tau = \omega_{r_o}\omega_{r_8}\sigma$ induces on each diagram Δ_J and $\Delta_J^{\perp}$ the unique non-trivial symmetry. Thus we have $\Omega_J = \langle W_J^{\perp},\tau\rangle$, $\tau^2 = 1$. Now the element $\omega_o' \in W_J^{\perp}$ maps r_1, r_2, r_3, r_4, r_5, r_6 to $-r_6$, $-r_2$, $-r_5$, $-r_4$, $-r_3$, $-r_1$ respectively. Multiplying the element τ on the left by ω_o', we obtain an element τ_o of Ω_J which induces also the non-trivial symmetry on Δ_J and commutes pointwise with $W_J^{\perp}$. Therefore, we have $\Omega_J = W(E_6) \times \{1,\tau_o\}$.

Now the conjugacy classes of Ω_J can be determined by the theory of admissible diagrams in [5]. There are 25 classes in $W(E_6)$ and so 50 classes in Ω_J. We represent each conjugacy class in $W(E_6)$ by an element s_i whose reduced expressions (in the sense of Carter's paper) can be read off from the following diagrams.



In the above diagrams we have put $a = \frac{1}{2}(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4+\varepsilon_5-\varepsilon_6-\varepsilon_7+\varepsilon_8)$ and $b = \frac{1}{2}(-\varepsilon_1-\varepsilon_2-\varepsilon_3+\varepsilon_4-\varepsilon_5-\varepsilon_6-\varepsilon_7+\varepsilon_8)$.

The other 25 classes can be determined now by the elements $s_i\tau_o$, $1 \le i \le 25$.

Now if $\overline{G_J^g}$ is an orbit in $\mathfrak{G}_J\backslash G_\sigma$ determined by a conjugacy class $[\omega]$ in Ω_J, as it is described in Proposition 4.1, then we have $|(G_J^g)_\sigma| = |(M^g)_\sigma||(S^g)_\sigma| = |M_{\sigma,\omega}||S_{\sigma,\omega}|$.

If $\omega = s_i$, $1 \le i \le 25$, then $|(M^g)_\sigma| = |A_2(q)|$. If $\omega = s_i\tau_o$, $1 \le i \le 25$ then $|(M^g)_\sigma| = |{}^2A_2(q^2)|$.

For the torus $(S^g)_\sigma$, we have $(S^g)_\sigma \simeq {}^{X/\overline{P}_J}/_{(q\omega-1)}\,\frac{X}{\overline{P}_J}$.

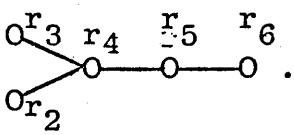
Thus $|(S^g)_\sigma| = f_{\omega,V/V_J}(q)$, where $f_{\omega,V/V_J}(t)$ is the characteristic polynomial of ω on V/V_J, $V = X \otimes \mathbb{R}$, $V_J = \overline{P}_J \otimes \mathbb{R} = P_J \otimes \mathbb{R}$.

Now $f_{\omega,V/V_J}(t) = \dfrac{f_{\omega,V}(t)}{f_{\omega,V_J}(t)} = f_{\omega,V_J^\perp}(t)$, where $V_J^\perp$ is the orthogonal subspace to V_J. If $\omega = s_i$, $1 \le i \le 25$, then $f_{\omega,V_J^\perp}(t)$ can be read off from the admissible diagram of s_i. If $\omega = s_i\tau_o$, $1 \le i \le 25$, then $f_{\omega,V_J^\perp}(t)$ can be obtained if in $f_{s_i,V_J^\perp}(t)$ we replace the factors $t^n + t^{n-1} + \ldots + 1$ by $t^n - t^{n-1} + \ldots (-1)^n$. This is because $\tau_o = -1$ on $V_J^\perp$.

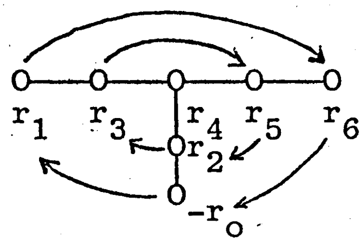
The second example will show now how the conditions on q are determined in the case of a non-parabolic subset.

Example 2: We consider the Dynkin diagram Δ_J: $\overset{r_1}{\circ}\!\!-\!\!\overset{r_3}{\circ}\quad \overset{r_2}{\circ}\!\!-\!\!\overset{-r_o}{\circ}\quad \overset{r_5}{\circ}\!\!-\!\!\overset{r_6}{\circ}$ of the extended Dynkin diagram of type E_6. Δ_J is a non-parabolic subset and we have to look for the conditions for occurrence of each family separately. Here we have $\Delta_J^\perp = \phi$ and so $\Omega_J \simeq \mathrm{Aut}_W(\Delta_J)$.

Let ω_o' be the element considered in the Example 1 and ω_o'' the element of maximal length in the Weyl group $W(D_5)$ with diagram



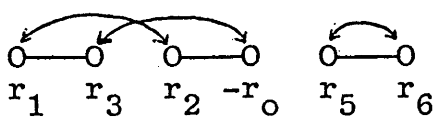
. Then the element $\sigma_1 = \omega_o'' \omega_o'$ induces the following symmetry on $\tilde{\Delta}$:



.

We see that $\sigma_1 \in \Omega_J$. Also we consider the element

$\sigma_2 = s_o' \, s_o \, \omega_{\varepsilon_4+\varepsilon_2} \, \omega_{\varepsilon_4-\varepsilon_2} \, \omega_{\varepsilon_5+\varepsilon_2}$, where s_o, s_o' are the elements of maximal lengths in the Weyl subgroups $W_1 = \langle \omega_{r_2}, \omega_{r_o} \rangle$, $W_2 = \langle \omega_{r_1}, \omega_{r_3} \rangle$ respectively. We see that the element σ_2 induces the symmetry:



on Δ_J and so $\sigma_2 \in \Omega_J$. Now the elements σ_1, σ_2 are not conjugate and generate a group isomorphic to the symmetric group S_3. Looking at the tables in [5] for the orders of $\mathrm{Aut}_W(\Delta_J)$ we deduce that $\Omega_J = \langle \sigma_1, \sigma_2 \rangle$. Thus we obtain three families, namely the families $(3A_2, 1)$, $(3A_2, [\sigma_1])$ and $(3A_2, [\sigma_2])$. For the conditions which need to be imposed on q for the occurrence of these families we have to consider the set of roots $\bar{\Phi}_J - \Phi_J$, in the groups $\Gamma_{J,\omega}$ according to the Theorem 4.4. We have $\bar{\Phi}_J - \Phi_J \equiv \{r_4, 2r_4\}$ mod P_J. We must now distinguish between the different isogeny types for G - G can be either simply connected or adjoint.

The simply-connected case. We have $|\frac{X}{P}| = 3$ and $|\frac{P}{P_J}| = 3$, therefore $|\frac{X}{P_J}| = 9$. Thus $\frac{X}{P_J}$ is either of type $Z_3 \oplus Z_3$ or of type Z_9. But since $3\lambda_i = 0$ in $\frac{X}{P_J}$ for all $i = 1,2,\ldots,6$, we must have $\frac{X}{P_J} \approx Z_3 \oplus Z_3$, where the λ_i's are the fundamental weights of the root system $\Phi = \Phi(E_6)$. In fact, if we put $v = \frac{1}{3}(r_1 + 2r_2 + r_5 + 2r_6)$, we have $X/P_J \approx Zr_4 \oplus Zv$. We see now that since the order of r_4 in X/P_J is 3, for a character to exist for each group $\Gamma_{J,\omega}$ which does not annihilate r_4 and $2r_4$ it is necessary and sufficient that $r_4 \neq 0$ and $2r_4 \neq 0$ (from which, one of the two is redundant as $3r_4 = 0$).

Now we pass to each individual group $\Gamma_{J,\omega}$ where we impose the additional relations $(q\omega-1)r_4 = 0$ and $(q\omega-1)v = 0$. For $\omega = 1$, we see that the only possible relation in $\Gamma_{J,1}$ which could imply $r_4 = 0$ is $(q-1)r_4 = 0$. Since $3r_4 = 0$, this is so if and only if $3 \nmid q-1$. Thus the condition we are looking for is $3|q-1$.

For $\omega = \sigma_1$, we have $(q\sigma_1-1)r_4 = (q-1)r_4 = 0$ and $(q\sigma_1-1)v = \frac{q-1}{3}(r_1+2r_2+r_5+2r_6) - 2qr_4 = 0$. We see here that the family $(3A_2,[\sigma_1])$ does not occur for any q. For, if r_4 were non-zero, then we would have $3|q-1$ from the first relation and so $3|q$ from the second one. This is absurd.

For $\omega = \sigma_2$, we have $(q\sigma_2-1)r_4 = (2q-1)r_4 = 0$ and $\frac{q-1}{3}(r_1 + 2r_2 + r_5 + 2r_6) = 2qr_4$. Thus $r_4 \neq 0$ if and only if $3|2q-1$ i.e. if and only if $3|q+1$. Hence the conditon here is $3|q+1$.

The above conditions on q can also be obtained easily by using the Brauer complex. For this we have to look for which q there is a solution of one of the sets of inequalities in (a)' and (b)' which satisfy the system of equations (*) of Chapter 4, pp. 50 , in Chapter 4.

For $\omega = 1$, q has to satisfy the equation $3\mu_4 = q-1$, for some $\mu_4 \in Z^+_{>0}$ such that $z_1 = 2\mu_4$, $z_2 = 3\mu_4$, $z_3 = 4\mu_4$, $z_4 = 6\mu_4$, $z_5 = 4\mu_4$, $z_6 = 2\mu_4$ are integers. This is so if and only if $3|q-1$.

For $\omega = \sigma_1$, q has to satisfy the equation $3\mu_4 = q-1$ for some $\mu_4 \in Z^+_{>0}$ such that $z_1 = 2\mu_4 + \frac{4}{3}$ is an integer. Such a μ_4 cannot exist. The same we find for the other element σ_1^{-1} in the class $[\sigma_1]$. Thus the family $(3A_2,[\sigma_1])$ does not exist for any q.

For $\omega = \sigma_2$, q has to satisfy the equation $3\mu_4 = q-2$ for some $\mu_4 \in Z^+_{>0}$ such that $z_1 = 2\mu_4 + \frac{5}{3}$ is an integer. Such a μ_4 cannot exist. So we have to look for the other elements in the class $[\sigma_2]$ of σ_2 in Ω_J. $\omega = \sigma_2\sigma_1$ is one of them. For this ω, q has to satisfy the equation $3\mu_4 = q-2$, for some $\mu_4 \in Z^+_{>0}$ such that $z_1 = 2\mu_4 + 1$, $z_2 = 3\mu_4 + 2$, $z_3=4\mu_4 + 2$, $z_4 = 6\mu_4 + 3$, $z_5 = 3\mu_4 + 3$, $z_6 = 2\mu_4 + 1$ are integers. This is so if and only if $3|q+1$.

The adjoint case. Detailed calculations show that the conditions on q here are the same as in the simply-connected case, except for the family $(3A_2,[\sigma_1])$ which occurs for all sufficiently large q which satisfy the condition $3|q-1$.

For the orders of the centralizers determined by the above three families we have to work in the same way as we did in Example 1.

In the tables which follow one row corresponds to each family. The 1st column gives the type E_J of the family $(E_J,[\omega])$. The 2nd column gives the abstract type of group which is isomorphic to Ω_J. The 3rd and 4th columns give respectively the orders of the semi-simple and toral parts of the centralizers determined by a family. In particular from the semi-simple part we can deduce what kind of graph automorphism is induced by ω. The 5th column gives the conditions which need to be imposed on q for the occurrence of the families. In this last column whenever there is no indication of condition for occurrence of a family this will mean that the subset Δ_J can be taken to be strictly parabolic subset and so the corresponding families $(E_J,[\omega])$ occur for all q sufficiently large. If Δ_J is a non-parabolic subset then for the condition for occurrence of the families $(E_J,[\omega])$ we shall distinguish the simply-connected case from the adjoint one by putting "sc" for the former and "ad" for the latter.

Our notation for the types of Δ_J's will be as that of Dynkin's paper [10]. Also we shall denote by H_1 and H_2 the following groups. Let us write the symmetric group S_4 as the semi-direct product of $Z_2 \times Z_2$ by S_3. Then H_1 denotes the semi-direct product of $Z_2 \times Z_2 \times Z_2$ by S_4, where S_4 acts on the normal part such that its Klein subgroup $Z_2 \times Z_2$ acts trivially and S_3 permutes the components in all possible ways. H_2 denotes the semi-direct product of the Weyl group of type D_4 by S_4, where here the Klein subgroup acts trivially on $W(D_4)$ and S_3 acts as in the figure:

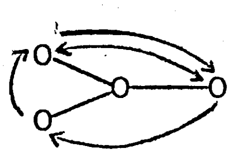


TABLE 2

Orders of the connected centralizers of semi-simple elements in $G_2(q)$.

Type of Δ_J	Ω_J	$\|(M^g)_\sigma\|$	$\|(S^g)_\sigma\|$	condition for occurrence
A_1	Z_2	$\|A_1(q)\|$	$q-1$	
			$q+1$	
$\tilde{A}_1$	Z_2	$\|A_1(q)\|$	$q-1$	
			$q+1$	
$A_1+\tilde{A}_1$	1	$\|A_1(q)\|^2$	1	$2\|q-1$
A_2	Z_2	$\|A_2(q)\|$	1	$3\|q-1$
		$\|{}^2A_2(q^2)\|$	1	$3\|q+1$
G_2	1	$\|G_2(q)\|$	1	

TABLE 3

Orders of the connected centralizers of semi-simple elements in $F_4(q)$.

Type of Δ_J	Ω_J	$\|(M^g)_\sigma\|$	$\|(S^g)_\sigma\|$	condition for occurrence
A_1	$W(B_3)$	$\|A_1(q)\|$	$(q-1)^3$	
			$(q+1)(q-1)^2$	
			$(q-1)(q+1)^2$	
			$(q+1)^3$	
			$(q+1)(q-1)^2$	
			$(q-1)(q+1)^2$	
			$(q-1)(q^2+1)$	
			$(q+1)(q^2+1)$	
			q^3-1	
			q^3+1	

TABLE 3 (Continued)

Type of Δ_J	Ω_J	$\|(M^g)_\sigma\|$	$\|(S^g)_\sigma\|$	condition for occurrence
$\tilde{A}_1$	$W(B_3)$	$\|A_1(q)\|$	$(q-1)^3$	
			$(q+1)(q-1)^2$	
			$(q-1)(q+1)^2$	
			$(q+1)^3$	
			$(q+1)(q-1)^2$	
			$(q-1)(q+1)^2$	
			$(q-1)(q^2+1)$	
			$(q+1)(q^2+1)$	
			q^3-1	
			q^3+1	
$2A_1$	$D_8 \times Z_2$	$\|A_1(q)\|^2$	$(q-1)^2$	$2\|q-1$
			q^2-1	$2\|q-1$
			$(q+1)^2$	$2\|q-1$
			q^2-1	$2\|q-1$
			q^2+1	$2\|q-1$
		$\|A_1(q^2)\|$	$(q-1)^2$	$2\|q-1$
			q^2-1	$2\|q-1$
			$(q+1)^2$	$2\|q-1$
			q^2-1	$2\|q-1$
			q^2+1	$2\|q-1$
$A_1+\tilde{A}_1$	$Z_2 \times Z_2$	$\|A_1(q)^2\|$	$(q-1)^2$	
			q^2-1	
			q^2-1	
			$(q+1)^2$	
A_2	$S_3 \times Z_2$	$\|A_2(q)\|$	$(q-1)^2$	
			q^2-1	
			q^2+q+1	
		$\|{}^2A_2(q^2)\|$	$(q+1)^2$	
			q^2-1	
			q^2-q+1	

TABLE 3 (Continued)

Type of Δ_J	Ω_J	$\vert(M^g)_\sigma\vert$	$\vert(S^g)_\sigma\vert$	condition for occurrence
$\tilde{A}_2$	$S_3 \times Z_2$	$\vert A_2(q)\vert$	$(q-1)^2$	
			q^2-1	
			q^2+q+1	
		$\vert {}^2A_2(q^2)\vert$	$(q+1)^2$	
			q^2-1	
			q^2-q+1	
B_2	D_8	$\vert B_2(q)\vert$	$(q-1)^2$	
			q^2-1	
			q^2-1	
			q^2+1	
			$(q+1)^2$	
$2A_1+\tilde{A}_1$	$Z_2 \times Z_2$	$\vert A_1(q)\vert^3$	$q-1$	$2\vert q-1$
			$q+1$	$2\vert q-1$
		$\vert A_1(q)\vert\,\vert A_1(q^2)\vert$	$q-1$	$2\vert q-1$
			$q+1$	$2\vert q-1$
$A_2+\tilde{A}_1$	Z_2	$\vert A_2(q)\vert\,\vert A_1(q)\vert$	$q-1$	
		$\vert {}^2A_2(q^2)\vert\,\vert A_1(q)\vert$	$q+1$	
$\tilde{A}_2+A_1$	Z_2	$\vert A_2(q)\vert\,\vert A_1(q)\vert$	$q-1$	
		$\vert {}^2A_2(q^2\vert\,\vert A_1(q)\vert$	$q+1$	
B_2+A_1	Z_2	$\vert B_2(q)\vert\,\vert A_1(q)\vert$	$q-1$	$2\vert q-1$
			$q+1$	$2\vert q-1$
A_3	$Z_2 \times Z_2$	$\vert A_3(q)\vert$	$q-1$	$2\vert q-1$
			$q+1$	$2\vert q-1$
		$\vert {}^2A_3(q^2)\vert$	$q-1$	$2\vert q-1$
			$q+1$	$2\vert q-1$

TABLE 3 (Continued)

Type of Δ_J	Ω_J	$\lvert (M^g)_\sigma \rvert$	$\lvert (S^g)_\sigma \rvert$	condition for occurrence
B_3	Z_2	$\lvert B_3(q) \rvert$	$q-1$	
			$q+1$	
C_3	Z_2	$\lvert C_3(q) \rvert$	$q-1$	
			$q+1$	
$A_2+\tilde{A}_2$	Z_2	$\lvert A_2(q) \rvert^2$	1	$3 \mid q-1$
		$\lvert {}^2A_2(q^2) \rvert^2$	1	$3 \mid q-1$
$A_3+\tilde{A}_1$	Z_2	$\lvert A_3(q) \rvert \lvert A_1(q) \rvert$	1	$4 \mid q-1$
		$\lvert {}^2A_3(q^2) \rvert \lvert A_1(q) \rvert$	1	$4 \mid q-1$
C_3+A_1	1	$\lvert C_3(q) \rvert \lvert A_1(q) \rvert$	1	$2 \mid q-1$
B_4	1	$\lvert B_4(q) \rvert$	1	$2 \mid q-1$
F_4	1	$\lvert F_4(q) \rvert$	1	

TABLE 4

Orders of the connected centralizers of semi-simple elements in $E_6(q)$

Type of Δ_J	Ω_J	$\lvert (M^g)_\sigma \rvert$	$\lvert (S^g)_\sigma \rvert$	condition for occurrence sc.	ad.
A_1	S_6	$\lvert A_1(q) \rvert$	$(q-1)^5$		
			$(q+1)(q-1)^4$		
			$(q+1)^2(q-1)^3$		
			$(q-1)^2(q^3-1)$		
			$(q-1)^2(q+1)^3$		
			$(q^2-1)(q^3-1)$		
			$(q-1)(q^4-1)$		
			$(q^2+q+1)(q^3-1)$		
			$(q+1)(q^4-1)$		
			q^5-1		
			$(q^2+q+1)(q^3+1)$		

TABLE 4 (Continued)

Type of Δ_J	Ω_J	$\lvert(M^g)_\sigma\rvert$	$\lvert(S^g)_\sigma\rvert$	condition for occurrence sc.	ad.
$2A_1$	$S_4 \times Z_2$	$\lvert A_1(q)\rvert^2$	$(q-1)^4$		
			$(q+1)(q-1)^3$		
			$(q^2-1)^2$		
			$(q-1)(q^3-1)$		
			q^4-1		
		$\lvert A_1(q^2)\rvert$	$(q+1)^2(q^2-1)$		
			$(q^2-1)^2$		
			$(q-1)^2(q^2-1)$		
			$(q-1)(q^3+1)$		
			$(q-1)^2(q^2+1)$		
A_2	$S_3 \wr Z_2$	$\lvert A_2(q)\rvert$	$(q-1)^4$		
			$(q+1)(q-1)^3$		
			$(q-1)(q^3-1)$		
			$(q^2-1)^2$		
			$(q+1)(q^3-1)$		
			$(q^2+q+1)^2$		
		$\lvert {}^2A_2(q^2)\rvert$	$(q^2-1)^2$		
			q^4-1		
			q^4+q^2+1		
$3A_1$	$S_3 \times Z_2$	$\lvert A_1(q)\rvert^3$	$(q-1)^3$		
			$(q+1)(q-1)^2$		
		$\lvert A_1(q^2)\rvert\lvert A_1(q)\rvert$	$(q+1)(q-1)^2$		
			$(q-1)(q+1)^2$		
		$\lvert A_1(q^3)\rvert$	q^3-1		
			$(q+1)(q^2+q+1)$		
A_2+A_1	S_3	$\lvert A_2(q)\rvert\lvert A_1(q)\rvert$	$(q-1)^3$		
			$(q+1)(q-1)^2$		
			q^3-1		

TABLE 4 (Continued)

Type of Δ_J	Ω_J	$\|(M^g)_\sigma\|$	$\|(S^g)_\sigma\|$	condition for occurrence sc.	ad.
A_3	D_8	$\|A_3(q)\|$	$(q-1)^3$		
			$(q+1)(q-1)^2$		
			$(q-1)(q+1)^2$		
		$\|{}^2A_3(q^2)\|$	$(q+1)(q-1)^2$		
			$(q-1)(q^2+1)$		
$4A_1$	S_4	$\|A_1(q)\|^4$	$(q-1)^2$	$2\|q-1$	$2\|q-1$
		$\|A_1(q^2)\|\|A_1(q)\|^2$	q^2-1	$2\|q-1$	$2\|q-1$
		$\|A_1(q^3)\|\|A_1(q)\|$	q^2+q+1	$2\|q-1$	$2\|q-1$
		$\|A_1(q^2)\|^2$	$(q-1)^2$	$2\|q-1$	$2\|q-1$
		$\|A_1(q^4)\|$	q^2-1	$2\|q-1$	$2\|q-1$
A_2+A_1	Z_2	$\|A_2(q)\|\|A_1(q)\|^2$	$(q-1)^2$		
		$\|{}^2A_2(q^2)\|\|A_1(q^2)\|$	q^2-1		
$2A_2$	$S_3 \times Z_2$	$\|A_2(q)\|^2$	$(q-1)^2$		
			(q^2-1)		
			q^2+q+1		
		$A_2(q^2)$	$(q+1)^2$		
			q^2-1		
			q^2-q+1		
A_3+A_1	Z_2	$\|A_3(q)\|\|A_1(q)\|$	$(q-1)^2$		
			q^2-1		
A_4	Z_2	$\|A_4(q)\|$	$(q-1)^2$		
			q^2-1		
D_4	S_3	$\|D_4(q)\|$	$(q-1)^2$		
		$\|{}^2D_4(q^2)\|$	q^2-1		
		$\|{}^3D_4(q^3)\|$	q^2+q+1		

TABLE 4 (Continued)

Type of Δ_J	Ω_J	$\|(M^g)_\sigma\|$	$\|(S^g)_\sigma\|$	condition for occurrence sc.	ad.
$2A_2+A_1$	Z_2	$\|A_2(q)\|^2\|A_1(q)\|$	q-1		
			q+1		
A_3+2A_1	Z_2	$\|A_3(q)\|\|A_1(q)\|^2$	q-1	$2\|q-1$	$2\|q-1$
		$\|{}^2A_3(q^2)\|\|A_1(q^2)\|$	q-1	$2\|q-1$	$2\|q-1$
A_4+A_1	1	$\|A_4(q)\|\|A_1(q)\|$	q-1		
A_5	Z_2	$\|A_5(q)\|$	q-1		
			q+1		
D_5	1	$\|D_5(q)\|$	q-1		
$3A_2$	S_3	$\|A_2(q)\|^3$	1	$3\|q-1$	$3\|q-1$
		$\|{}^2A_2(q^2)\|\|A_2(q^2)\|$	1	$3\|q+1$	$3\|q+1$
		$\|A_3(q^3)\|$	1	never occurs	$3\|q-1$
A_5+A_1	1	$\|A_5(q)\|\|A_1(q)\|$	1	$2\|q-1$	$2\|q-1$
E_6	1	$\|E_6(q)\|$	1		

TABLE 5

Orders of the connected centralizers of semi-simple elements in $E_7(q)$.

Type of Δ_J	Ω_J	$\lvert (M^g)_\sigma \rvert$	$\lvert (S^g)_\sigma \rvert$	condition for occurrence sc.	ad.
A_1	$W(D_6)$	$\lvert A_1(q) \rvert$	$(q-1)^6$		
			$(q^2-1)(q-1)^4$		
			$(q^3-1)(q-1)^3$		
			$(q-1)^2(q^2-1)^2$		
			$(q-1)^2(q^4-1)$		
			$(q-1)(q^2-1)(q^3-1)$		
			$(q-1)(q^5-1)$		
			$(q^2-1)^3$		
			$(q^2-1)^3$		
			$(q^2-1)(q^4-1)$		
			$(q^2-1)(q^4-1)$		
			$(q^3-1)^2$		
			q^6-1		
			q^6-1		
			$(q+1)^2(q-1)^4$		
			$(q+1)^2(q^2-1)^2$		
			$(q+1)^6$		
			$(q^2-1)^3$		
			$(q-1)(q+1)^5$		
			$(q+1)^2(q^4-1)$		
			$(q-1)^2(q^4-1)$		
			$(q+1)(q^2-1)(q^3-1)$		
			$(q+1)^3(q^3+1)$		
			$(q-1)(q^2-1)(q^3+1)$		
			$(q+1)(q^2-1)^2$		
			$(q^2+1)(q^2-1)^2$		

TABLE 5 (continued)

Type of Δ_J	Ω_J	$\lvert(M^g)_\sigma\rvert$	$\lvert(S^g)_\sigma\rvert$	condition for occurrence sc.	ad.
A_1	$W(D_6)$	$\lvert A_1(q)\rvert$	$(q-1)^2(q^2+1)^2$		
			$(q+1)^2(q^2+1)^2$		
			$(q^2-1)(q^4+1)$		
			$(q^2+q+1)(q^4-1)$		
			$(q+1)(q^2-1)(q^3+1)$		
			$(q-1)(q^2+1)(q^3+1)$		
			$(q+1)(q^5+1)$		
			$(q^2+1)(q^4-1)$		
			$(q^2+1)(q^4+1)$		
			$(q^3+1)^2$		
$2A_1$	$W(B_4)\times Z_2$	$\lvert A_1(q)\rvert^2$	$(q-1)^5$		
			$(q^2-1)(q-1)^3$		
			$(q-1)(q^2-1)^2$		
			$(q-1)(q^2-1)^2$		
			$(q-1)^2(q^3-1)$		
			$(q-1)(q^4-1)$		
			$(q-1)(q^4-1)$		
			$(q^2-1)(q+1)^3$		
			$(q+1)(q^2-1)^2$		
			$(q^2-1)(q^3+1)$		
			$(q-1)(q^2+1)^2$		
			$(q^2-1)(q-1)^3$		
			$(q-1)(q^2-1)^2$		
			$(q+1)(q^2-1)^2$		
			$(q+1)(q^2-1)^2$		
			$(q^2-1)(q^3-1)$		

TABLE 5 (contined)

Type of Δ_J	Ω_J	$\|(M^g)_\sigma\|$	$\|(S^g)_\sigma\|$	condition for occurrence sc. ad.
$2A_1$	$W(B_4) \times Z_2$	$\|A_1(q)\|^2$	$(q+1)(q^4-1)$	
			$(q+1)(q^4-1)$	
			$(q+1)^5$	
			$(q-1)(q+1)^4$	
			$(q+1)^2(q^3+1)$	
			$(q+1)(q^2+1)^2$	
		$\|A_1(q^2)\|$	$(q-1)(q^2-1)^2$	
			$(q-1)(q^4-1)$	
			$(q+1)(q^2-1)^2$	
			$(q^2-1)(q+1)^3$	
			$(q^2-1)(q^3+1)$	
			$(q+1)^2(q^3-1)$	
			$(q+1)(q^4-1)$	
			$(q+1)(q^4+1)$	
			$(q^2+1)(q+1)^3$	
			$(q^2-1)(q-1)^3$	
			$(q^2+1)(q-1)^3$	
			$(q-1)(q^2-1)^2$	
			$(q+1)(q^2-1)^2$	
			$(q-1)^2(q^3+1)$	
			$(q^2-1)(q^3-1)$	
			$(q-1)(q^4-1)$	
			$(q-1)(q^4+1)$	
			$(q+1)(q^4-1)$	
A_2	$S_6 \times Z_2$	$\|A_2(q)\|$	$(q-1)^5$	
			$(q+1)(q-1)^4$	
			$(q-1)(q^2-1)^2$	

TABLE 5 (continued)

Type of Δ_J	Ω_J	$\lvert(M^g)_\sigma\rvert$	$\lvert(S^g)_\sigma\rvert$	condition for occurrence sc.	ad.
A_2	$S_6 \times Z_2$	$\lvert A_2(q)\rvert$	$(q-1)^2(q^3-1)$		
			$(q-1)^2(q+1)^3$		
			$(q^2-1)(q^3-1)$		
			$(q-1)(q^4-1)$		
			$(q^2+q+1)(q^3-1)$		
			$(q+1)(q^4-1)$		
			(q^5-1)		
			$(q^2+q+1)(q^3+1)$		
		$\lvert {}^2A_2(q^2)\rvert$	$(q+1)^5$		
			$(q^2-1)(q+1)^3$		
			$(q+1)(q^2-1)^2$		
			$(q+1)^2(q^3+1)$		
			$(q-1)(q^2-1)^2$		
			$(q^2-1)(q^3+1)$		
			$(q+1)(q^4-1)$		
			$(q^2-q+1)(q^3+1)$		
			$(q-1)(q^4-1)$		
			(q^5+1)		
			$(q^2-q+1)(q^3-1)$		
$[3A_1]'$	$W(B_3) \times Z_2$	$\lvert A_1(q)\rvert^3$	$(q-1)^4$		
			$(q^2-1)(q-1)^2$		
			$(q^2-1)(q-1)^2$		
			$(q^2-1)^2$		
			$(q^2-1)^2$		
			$(q^2-1)(q+1)^2$		
			$(q^2-1)(q+1)^2$		
			$(q+1)^4$		

TABLE 5 (continued)

Type of Δ_J	Ω_J	$\lvert(M^g)_\sigma\rvert$	$\lvert(S^g)_\sigma\rvert$	condition for occurrence sc.	ad.
$[3A_1]'$	$W(B_3)\times Z_2$	$\lvert A_1(q)\rvert\lvert A_1(q^2)\rvert$	$(q^2-1)(q-1)^2$		
			$(q^2-1)^2$		
			$(q^2-1)^2$		
			$(q^2-1)^2$		
			$(q^2-1)(q+1)^2$		
			$(q^2-1)(q+1)^2$		
			$(q^2-1)(q+1)^2$		
			$(q^2+1)(q+1)^2$		
		$\lvert A_1(q^3)\rvert$	$(q-1)(q^3-1)$		
			$(q+1)(q^3-1)$		
			$(q-1)(q^3+1)$		
			$(q+1)(q^3+1)$		
$[3A_1]''$	$W(F_4)$	$\lvert A_1(q)\rvert^3$	$(q-1)^4$		
			$(q+1)(q-1)^3$		
			$(q^2-1)^2$		
			$(q-1)(q^3-1)$		
			(q^4-1)		
			$(q+1)^4$		
			$(q-1)(q+1)^3$		
			$(q+1)(q^3+1)$		
			$(q^2+1)^2$		
		$\lvert A_1(q)\rvert\lvert A_1(q^2)\rvert$	$(q^2-1)(q-1)^2$		
			$(q^2+1)(q-1)^2$		
			$(q^2-1)^2$		
			$(q^2-1)(q+1)^2$		
			$(q-1)(q^3+1)$		

TABLE 5 (continued)

Type of Δ_J	Ω_J	$\|(M^g)_\sigma\|$	$\|(S^g)_\sigma\|$	condition for occurrence sc.	ad.
$[3A_1]''$	$W(F_4)$	$\|A_1(q)\|\|A_1(q^2)\|$	$(q+1)(q^3-1)$		
			(q^4-1)		
			q^4+1		
			$(q^2+1)(q+1)^2$		
		$\|A_1(q^3)\|$	$(q-1)(q^3-1)$		
			$(q-1)(q^3+1)$		
			$(q+1)(q^3-1)$		
			$(q+1)(q^3+1)$		
			(q^4-q^2+1)		
			$(q^2-q+1)^2$		
			$(q^2+q+1)^2$		
$A_2 + A_1$	$S_4 \times Z_2$	$\|A_1(q)\|\|A_2(q)\|$	$(q-1)^4$		
			$(q+1)(q-1)^3$		
			$(q-1)(q^3-1)$		
			$(q^2-1)^2$		
			(q^4-1)		
		$\|A_1(q)\|\|{}^2A_2(q^2)\|$	$(q+1)^4$		
			$(q+1)^2(q^2-1)$		
			$(q+1)(q^3+1)$		
			$(q^2-1)^2$		
			q^4-1		
A_3	$S_4 \times Z_2 \times Z_2$	$\|A_3(q)\|$	$(q-1)^4$		
			$(q+1)(q-1)^3$		
			$(q-1)(q^3-1)$		
			$(q^2-1)^2$		
			q^4-1		

TABLE 5 (continued)

Type Δ_J	Ω_J	$\|(M^g)_\sigma\|$	$\|(S^g)_\sigma\|$	condition for occurrence sc.	ad.
A_3	$S_4 \times Z_2 \times Z_2$	$\|A_3(q)\|$	$(q+1)(q-1)^3$		
			$(q^2-1)^2$		
			$(q+1)(q^3-1)$		
			$(q^2-1)(q+1)^2$		
			$(q^2+1)(q+1)^2$		
		$\|{}^2A_3(q^2)\|$	$(q+1)^2(q^2-1)$		
			$(q^2-1)^2$		
			$(q^3+1)(q-1)$		
			$(q-1)^2(q^2-1)$		
			$(q-1)^2(q^2+1)$		
			$(q+1)^4$		
			$(q+1)^2(q^2-1)$		
			$(q+1)(q^3+1)$		
			$(q^2-1)^2$		
			q^4-1		
$[4A_1]'$	H_1	$\|A_1(q)\|^4$	$(q-1)^3$	$2\|q-1$	$2\|q-1$
			$(q-1)^2(q^2-1)$	$2\|q-1$	$2\|q-1$
			$(q+1)(q^2-1)$	$2\|q-1$	$2\|q-1$
			$(q+1)^3$	$2\|q-1$	$2\|q-1$
		$\|A_1(q)\|^2\|A_1(q^2)\|$	$(q-1)(q^2-1)$	$2\|q-1$	$2\|q-1$
			$(q+1)(q^2-1)$	$2\|q-1$	$2\|q-1$
			$(q-1)(q^2+1)$	$2\|q-1$	$2\|q-1$
			$(q+1)(q^2+1)$	$2\|q-1$	$2\|q-1$
		$\|A_1(q)\|\|A_1(q^3)\|$	q^3-1	$2\|q-1$	$2\|q-1$
			q^3+1	$2\|q-1$	$2\|q-1$
		$\|A_1(q^4)\|$	$(q-1)(q^2-1)$	$2\|q-1$	$2\|q-1$

TABLE 5 (continued)

Type Δ'_J	Ω_J	$\lvert(M^g)_\sigma\rvert$	$\lvert(S^g)_\sigma\rvert$	condition for occurrence sc.	ad.
$[4A_1]'$	H_1	$\lvert A_1(q^4)\rvert$	$(q-1)(q^2+1)$	$2\mid q-1$	$2\mid q-1$
			$(q+1)(q^2-1)$	$2\mid q-1$	$2\mid q-1$
			$(q+1)(q^2+1)$	$2\mid q-1$	$2\mid q-1$
		$\lvert A_1(q^2)\rvert^2$	$(q-1)^3$	$2\mid q-1$	$2\mid q-1$
			$(q-1)(q^2-1)$	$2\mid q-1$	$2\mid q-1$
			$(q-1)(q^2-1)$	$2\mid q-1$	$2\mid q-1$
			$(q+1)(q^2-1)$	$2\mid q-1$	$2\mid q-1$
			$(q+1)(q^2-1)$	$2\mid q-1$	$2\mid q-1$
			$(q+1)^3$	$2\mid q-1$	$2\mid q-1$
$[4A_1]''$	$W(B_3)$	$\lvert A_1(q)\rvert^4$	$(q-1)^3$		
			$(q+1)(q-1)^2$		
			$(q+1)(q^2-1)$		
			$(q+1)^3$		
		$\lvert A_1(q)\rvert\lvert A_1(q^3)\rvert$	q^3-1		
			q^3+1		
			$(q-1)(q^2-1)$		
			$(q+1)(q^2-1)$		
			$(q-1)(q^2+1)$		
			$(q+1)(q^2+1)$		
$A_2 + 2A_1$	$Z_2\times Z_2\times Z_2$	$\lvert A_2(q)\rvert\lvert A_1(q)\rvert^2$	$(q-1)^3$		
			$(q-1)(q^2-1)$		
		$\lvert A_2(q)\rvert\lvert A_1(q^2)\rvert$	$(q-1)(q^2-1)$		
			$(q+1)(q^2-1)$		
		$\lvert {}^2A_2(q^2)\rvert\lvert A_1(q)\rvert^2$	$(q+1)(q^2-1)$		
			$(q+1)^3$		
		$\lvert {}^2A_2(q^2)\rvert\lvert A_1(q^2)\rvert$	$(q-1)(q^2-1)$		

TABLE 5 (continued)

Type Δ_J	Ω_J	$\lvert(M^g)_\sigma\rvert$	$\lvert(S^g)_\sigma\rvert$	condition for occurrence sc.	ad.
$A_2 + 2A_1$	$Z_2 \times Z_2 \times Z_2$	$\lvert{}^2A_2(q^2)\rvert\lvert A_1(q^2)\rvert$	$(q+1)(q^2-1)$		
$2A_2$	$S_3 \times Z_2 \times Z_2$	$\lvert A_2(q)\rvert^2$	$(q-1)^3$		
			$(q+1)(q-1)^2$		
			q^3-1		
		$\lvert{}^2A_2(q^2)\rvert^2$	$(q+1)^3$		
			$(q+1)(q^2-1)$		
			q^3+1		
		$\lvert A_2(q^2)\rvert$	$(q-1)(q+1)^2$		
			$(q-1)(q^2-1)$		
			$(q^2-q+1)(q-1)$		
			$(q+1)(q-1)^2$		
			$(q+1)(q^2-1)$		
			$(q^2+q+1)(q+1)$		
$[A_3\ A_1]''$	$Z_2 \times Z_2 \times Z_2$	$\lvert A_3(q)\rvert\lvert A_1(q)\rvert$	$(q-1)^3$		
			$(q+1)(q-1)^2$		
			$(q+1)(q-1)^2$		
			$(q-1)(q+1)^2$		
		$\lvert{}^2A_3(q^2)\rvert\lvert A_1(q)\rvert$	$(q-1)(q^2-1)$		
			$(q+1)(q^2-1)$		
			$(q+1)(q^2-1)$		
			$(q+1)^3$		
$[A_3+A_1]''$	$S_4 \times Z_2$	$\lvert A_3(q)\rvert\lvert A_1(q)\rvert$	$(q-1)^3$		
			$(q+1)(q-1)^2$		
			$(q+1)(q^2-1)$		
			$(q+1)(q^2+1)$		
			q^3-1		

TABLE 5 (continued)

Type Δ_J	Ω_J	$\lvert(M^g)_\sigma\rvert$	$\lvert(S^g)_\sigma\rvert$	condition for occurrence sc.	ad.
$[A_3+A_1]''$	$S_4 \times Z_2$	$\lvert{}^2A_3(q^2)\rvert\,\lvert A_1(q)\rvert$	$(q-1)(q^2-1)$		
			$(q-1)(q^2+1)$		
			$(q+1)(q^2-1)$		
			$(q+1)^3$		
			q^3+1		
A_4	$S_4 \times Z_2$	$\lvert A_4(q)\rvert$	$(q-1)^3$		
			$(q+1)(q-1)^2$		
			q^3-1		
		$\lvert{}^2A_4(q^2\rvert$	$(q+1)^3$		
			$(q+1)(q^2-1)$		
			q^3+1		
D_4	$W(B_3)$	$\lvert D_4(q)\rvert'$	$(q-1)^3$		
			$(q+1)(q-1)^2$		
			$(q+1)(q^2-1)$		
			$(q+1)^3$		
		$\lvert{}^2D_4(q^2)\rvert$	$(q+1)(q^2-1)$		
			$(q+1)(q^2+1)$		
			$(q-1)(q^2+1)$		
			$(q-1)(q^2-1)$		
		$\lvert{}^3D_4(q^3)\rvert$	q^3-1		
			q^3+1		
$5A_1$	$(Z_2\times Z_2)\wr Z_2$	$\lvert A_1(q)\rvert^5$	$(q-1)^2$	$2\mid q-1$	$2\mid q-1$
			q^2-1	$2\mid q-1$	$2\mid q-1$
			$(q+1)^2$	$2\mid q-1$	$2\mid q-1$

TABLE 5 (continued)

Type Δ_J	Ω_J	$\|(M^g)_\sigma\|$	$\|(S^g)_\sigma\|$	condition for occurrence sc.	ad.
$5A_1$	$(Z_2 \times Z_2) \wr Z_2$	$\|A_1(q^2)\|\|A_1(q)\|^3$	q^2-1	$2\|q-1$	$2\|q-1$
			q^2+1	$2\|q-1$	$2\|q-1$
		$\|A_1(q^4)\|\|A_1(q)\|$	q^2-1	never occurs	$2\|q-1$
			q^2+1	never occurs	$2\|q-1$
		$\|A_1(q^2)\|^2\|A_1(q)\|$	$(q-1)^2$	never occurs	$2\|q-1$
			q^2-1	never occurs	$2\|q-1$
			q^2-1	never occurs	$2\|q-1$
			$(q+1)^2$	never occurs	$2\|q-1$
			$(q-1)^2$	$2\|q-1$	$2\|q-1$
			q^2-1	$2\|q-1$	$2\|q-1$
			$(q+1)^2$	$2\|q-1$	$2\|q-1$
A_2+3A_1	$S_3 \times Z_2$	$\|A_2(q)\|\|A_1(q)\|^3$	$(q-1)^2$		
		$\|A_2(q)\|\|A_1(q^2)\|\|A_1(q)\|$	q^2-1		
		$\|{}^2A_2(q^2)\|\|A_1(q^2)\|\|A_1(q)\|$	q^2-1		
		$\|{}^2A_2(q^2)\|\|A_1(q^3)\|$	q^2-1		
		$\|{}^2A_2(q^2)\|\|A_1(q)\|^3$	$(q+1)^2$		
		$\|A_2(q)\|\|A_1(q^3)\|$	q^2+q+1		
$2A_2+A_1$	$Z_2 \times Z_2$	$\|A_2(q)\|^2\|A_1(q)\|$	$(q-1)^2$		
		$\|A_2(q^2)\|\|A_1(q)\|$	q^2-1		
		$\|A_2(q^2)\|\|A_1(q)\|$	q^2-1		
		$\|{}^2A_2(q^2)\|^2\|A_1(q)\|$	$(q+1)^2$		
$[A_3+2A_1]'$	$Z_2 \times Z_2 \times Z_2$	$\|A_3(q)\|\|A_1(q)\|^2$	$q-1)^2$	$2\|q-1$	$2\|q-1$
			$(q^2-1$	$2\|q-1$	$2\|q-1$

TABLE 5 (continued)

Type Δ_J	Ω_J	$\lvert(M^g)_\sigma\rvert$	$\lvert(S^g)_\sigma\rvert$	condition for occurrence sc.	ad.
$[A_3+2A_1]'$	$Z_2\times Z_2\times Z_2$	$\lvert A_3(q)\rvert\lvert A_1(q^2)\rvert$	$(q+1)^2$	$2\mid q-1$	$2\mid q-1$
			q^2-1	$2\mid q-1$	$2\mid q-1$
		$\lvert {}^2A_3(q^2)\rvert\lvert A_1(q^2)\rvert$	$(q-1)^2$	$2\mid q-1$	$2\mid q-1$
			q^2-1	$2\mid q-1$	$2\mid q-1$
		$\lvert {}^2A_3(q^2)\rvert\lvert A_1(q)\rvert^2$	$(q+1)^2$	$2\mid q-1$	$2\mid q-1$
			q^2-1	$2\mid q-1$	$2\mid q-1$
$[A_2+2A_1]''$	$Z_2\times Z_2$	$\lvert A_3(q)\rvert\lvert A_1(q)\rvert^2$	$(q-1)^2$		
			q^2-1		
		$\lvert {}^2A_3(q^2)\rvert\lvert A_1(q)\rvert^2$	$(q+1)^2$		
			q^2-1		
A_3+A_2	$Z_2\times Z_2$	$\lvert A_3(q)\rvert\lvert A_2(q)\rvert$	$(q-1)^2$		
			q^2-1		
		$\lvert {}^2A_3(q^2)\rvert\lvert {}^2A_2(q)\rvert$	$(q+1)^2$		
			q^2-1		
A_4+A_1	Z_2	$\lvert A_4(q)\rvert\lvert A_1(q)\rvert$	$(q-1)^2$		
		$\lvert {}^2A_4(q^2)\rvert\lvert A_1(q)\rvert$	$(q+1)^2$		
$[A_5]'$	$Z_2\times Z_2$	$\lvert A_5(q)\rvert$	$(q-1)^2$		
			$q+1)^2$		
		$\lvert {}^2A_5(q^2)\rvert$	$(q+1)^2$		
			q^2-1		
$[A_5]''$	$S_3\times Z_2$	$\lvert A_5(q)\rvert$	$(q-1)^2$		
			q^2-1		
			q^2+q+1		
	$S_3\times Z_2$	$\lvert {}^2A_5(q^2)\rvert$	$(q+1)^2$		
			q^2-1		
			q^2-q+1		

TABLE 5 (continued)

Type Δ_J	Ω_J	$\vert(M^g)_\sigma\vert$	$\vert(S^g)_\sigma\vert$	condition for occurrence sc.	ad.
D_4+A_1	D_8	$\vert D_4(q)\vert\vert A_1(q)\vert$	$(q-1)^2$		
			q^2-1		
			$(q+1)^2$		
		$\vert{}^2D_4(q^2)\vert\vert A_1(q)\vert$	q^2-1		
			q^2+1		
D_5	$Z_2\times Z_2$	$\vert D_5(q)\vert$	$(q-1)^2$		
			q^2-1		
		$\vert{}^2D_5(q^2)\vert$	$(q+1)^2$		
			q^2-1		
$3A_2$	$S_3\times Z_2$	$\vert A_2(q)\vert^3$	$q-1$	$3\vert q-1$	$3\vert q-1$
		$\vert{}^2A_2(q^2)\vert^3$	$q+1$	$3\vert q+1$	$3\vert q+1$
		$\vert A_2(q^2)\vert\vert A_2(q)\vert$	$q+1$	$3\vert q-1$	$3\vert q-1$
		$\vert A_2(q^3)\vert$	$q-1$	$3\vert q-1$	$3\vert q-1$
		$\vert{}^2A_2(q^2)\vert\vert A_2(q^2)\vert$	$q-1$	$3\vert q+1$	$3\vert q+1$
		$\vert{}^2A_2(q^6)\vert$	$q+1$	$3\vert q+1$	$3\vert q+1$
A_3+3A_1	$Z_2\times Z_2$	$\vert A_3(q)\vert\vert A_1(q)\vert^3$	$q-1$	$2\vert q-1$	$2\vert q-1$
		$\vert{}^2A_3(q^2)\vert\vert A_1(q^2)\vert\vert A_1(q)\vert$	$q-1$	$2\vert q-1$	$2\vert q-1$
		$\vert{}^2A_3(q^2)\vert\vert A_1(q)\vert^3$	$q+1$	$2\vert q-1$	$2\vert q-1$
		$\vert A_3(q)\vert\vert A_1(q^2)\vert\vert A_1(q)\vert$	$q+1$	$2\vert q-1$	$2\vert q-1$
$A_3+A_2+A_1$	Z_2	$\vert A_3(q)\vert\vert A_2(q)\vert\vert A_1(q)\vert$	$q-1$		
		$\vert{}^2A_3(q^2)\vert\vert{}^2A_2(q^2)\vert\vert A_1(q)\vert$	$q+1$		
$2A_3$	$Z_2\times Z_2\times Z_2$	$\vert A_3(q)\vert^2$	$q-1$	$4\vert q-1$	$2\vert q-1$
			$q+1$	$4\vert q-1$	$2\vert q-1$
		$\vert A_3(q^2)\vert$	$q-1$	$4\vert q+1$	$2\vert q-1$
			$q+1$	$4\vert q+1$	$2\vert q-1$
			$q-1$	$4\vert q-1$	$2\vert q-1$
			$q+1$	$4\vert q-1$	$2\vert q-1$

TABLE 5 (continued)

Type Δ_J	Ω_J	$\|(M^g)_\sigma\|$	$\|(S^g)_\sigma\|$	condition for occurrence sc.	ad.
$2A_3$	$Z_2\times Z_2\times Z_2$	$\|{}^2A_3(q^2)\|^2$	q-1	$4\|q+1$	$2\|q-1$
			q+1	$4\|q+1$	$2\|q-1$
A_4+A_2	Z_2	$\|A_4(q)\|\,\|A_2(q)\|$	q-1		
		$\|{}^2A_4(q^2)\|\,\|{}^2A_2(q^2)\|$	q+1		
$[A_5+A_1]'$	Z_2	$\|A_5(q)\|\,\|A_1(q)\|$	q-1	$2\|q-1$	$2\|q-1$
		$\|{}^2A_5(q^2)\|\,\|A_1(q)\|$	q+1	$2\|q-1$	$2\|q-1$
$[A_5+A_1]''$	Z_2	$\|A_5(q)\|\,\|A_1(q)\|$	q-1		
		$\|{}^2A_5(q^3\|\,\|A_1(q)\|$	q+1		
A_6	Z_2	$\|A_6(q)\|$	q-1		
		$\|{}^2A_6(q^3\|$	q+1		
D_4+2A_1	$Z_2\times Z_2$	$\|D_4(q)\|\,\|A_1(q)\|^2$	q-1	$2\|q-1$	$2\|q-1$
			q+1	$2\|q-1$	$2\|q-1$
		$\|{}^2D_4(q^2)\|\,\|A_1(q^2)\|$	q-1	never occurs	$2\|q-1$
			q+1	never occurs	$2\|q-1$
D_5+A_1	Z_2	$\|D_5(q)\|\,\|A_1(q)\|$	q-1		
		$\|{}^2D_5(q^2)\|\,\|A_1(q)\|$	q+1		
D_6	Z_2	$\|D_6(q)\|$	q-1		
			q+1		
E_6	Z_2	$\|E_6(q)\|$	q-1		
		$\|{}^2E_6(q^2)\|$	q+1		
$2A_3+A_1$	$Z_2\times Z_2$	$\|A_3(q)\|^2\|A_1(q)\|$	1	$4\|q-1$	$4\|q-1$
		$\|A_3(q^2)\|\,\|A_1(q)\|$	1	never occurs	$4\|q-1$
			1	never occurs	$4\|q-1$
		$\|{}^2A_3(q^2)\|^2\|A_1(q)\|$	1	$4\|q+1$	$4\|q+1$

TABLE 5 (continued)

Type Δ_J	Ω_J	$\lvert(M^g)_\sigma\rvert$	$\lvert(S^g)_\sigma\rvert$	condition for occurrence sc.	ad.
A_5+A_2	Z_2	$\lvert A_5(q)\rvert\lvert A_2(q)\rvert$	1	$3\mid q-1$	$3\mid q-1$
		$\lvert {}^2A_5(q^2)\rvert\lvert {}^2A_2(q^2)\rvert$	1	$3\mid q+1$	$3\mid q+1$
A_7	Z_2	$A_7(q)$	1	$4\mid q-1$	$4\mid q-1$
		$\lvert {}^2A_7(q^2)\rvert$	1	$4\mid q+1$	$4\mid q+1$
D_6+A_1	1	$\lvert D_6(q)\rvert\lvert A_1(q)\rvert$	1	$2\mid q-1$	$2\mid q-1$
E_7	1	$\lvert E_7(q)\rvert$	1		

TABLE 6

Orders of the connected centralizers of semi-simple elements in $E_8(q)$.

Type of Δ_J	Ω_J	$\lvert(M^g)_\sigma\rvert$	$\lvert(S^g)_\sigma\rvert$	condition for occurrence
A_1	$W(E_7)$	$\lvert A_1(q)\rvert$	$(q-1)^7$	
			$(q+1)(q-1)^6$	
			$(q+1)^2(q-1)^5$	
			$(q^3-1)(q-1)^4$	
			$(q-1)(q^2-1)^3$	
			$(q-1)(q^2-1)^3$	
			$(q-1)^2(q^2-1)(q^3-1)$	
			$(q-1)^3(q^4-1)$	
			$(q+1)(q^2-1)^3$	
			$(q+1)(q^2-1)^3$	
			$(q^3-1)(q^2-1)^2$	
			$(q-1)(q^3-1)^2$	
			$(q-1)(q^2-1)(q^4-1)$	
			$(q-1)(q^2-1)(q^4-1)$	
			$(q-1)^2(q^5-1)$	
			$(q+1)(q^3+1)(q-1)^3$	
			$(q-1)^3(q^2+1)^2$	
			$(q-1)^2(q+1)^5$	
			$(q+1)^2(q^2-1)(q^3-1)$	
			$(q+1)(q^3-1)^2$	
			$(q+1)(q^2-1)(q^4-1)$	
			$(q+1)(q^2-1)(q^4-1)$	
			$(q^3-1)(q^4-1)$	
			$(q^2-1)(q^5-1)$	

TABLE 6 (continued)

Type of Δ_J	Ω_J	$\lvert (M^g)_\sigma \rvert$	$\lvert (S^g)_\sigma \rvert$	condition for occurrence
A_1	$W(E_7)$	$\lvert A_1(q) \rvert$	$(q-1)(q^6-1)$	
			$(q-1)(q^6-1)$	
			$(q^3+1)(q^2-1)^2$	
			$(q-1)(q^2+1)(q^4-1)$	
			$(q+1)(q-1)^2(q^4+1)$	
			$(q-1)^2(q^2+1)(q^3+1)$	
			$(q^2-1)(q+1)^5$	
			$(q-1)(q^2+q+1)^3$	
			$(q-1)(q^2+1)(q+1)^4$	
			$(q+1)^2(q^2+1)(q^3-1)$	
			$(q+1)(q^2+1)(q^4-1)$	
			$(q^2+q+1)(q^5-1)$	
			$(q+1)(q^6-1)$	
			$(q+1)(q^6-1)$	
			q^7-1	
			$(q^2-1)(q+1)^2(q^3+1)$	
			$(q+1)(q^2-1)(q^4+1)$	
			$(q^3+1)(q^4-1)$	
			$(q^2-1)(q^5+1)$	
			$(q-1)(q^2+1)(q^4+1)$	
			$(q-1)(q^3+1)^2$	
			$(q^3-1)(q^4-q^2+1)$	
			$(q-1)(q^6+q^3+1)$	
			$(q^3-1)(q^2-q+1)^2$	
			$(q+1)^7$	
			$(q+1)^3(q^2+1)^2$	
			$(q^3+1)(q^2+q+1)^2$	

TABLE 6 (continued)

Type of Δ_J	Ω_J	$\|(M^g)_\sigma\|$	$\|(S^g)_\sigma\|$	condition for occurrence
A_1	$W(E_7)$	$\|A_1(q)\|$	$(q+1)(q^2+1)(q^4+1)$	
			$(q^3+1)(q+1)^4$	
			$(q+1)^2(q^5+1)$	
			$(q+1)(q^3+1)^2$	
			$(q+1)(q^6-q^3+1)$	
			q^7+1	
			$(q^3+1)(q^4-q^2+1)$	
			$(q^2-q+1)(q^5+1)$	
			$(q+1)(q^2-q+1)^3$	
$2A_1$	$W(B_6)$	$\|A_1(q)\|^2$	$(q-1)^6$	
			$(q+1)(q-1)^5$	
			$(q+1)^2(q-1)^4$	
			$(q+1)^2(q-1)^4$	
			$(q^3-1)(q-1)^3$	
			$(q^2-1)^3$	
			$(q^2-1)^3$	
			$(q-1)(q^2-1)(q^3-1)$	
			$(q-1)^2(q^4-1)$	
			$(q-1)^2(q^4-1)$	
			$(q+1)^2(q^2-1)^2$	
			$(q+1)^2(q^2-1)^2$	
			$(q-1)(q+1)^2(q^3-1)$	
			$(q^3-1)^2$	
			$(q^2-1)(q^4-1)$	
			$(q^2-1)(q^4-1)$	
			$(q-1)(q^5-1)$	
			$(q-1)(q^2-1)(q^3+1)$	

TABLE 6 (continued)

Type of Δ_J	Ω_J	$\lvert (M^g)_\sigma \rvert$	$\lvert (S^g)_\sigma \rvert$	condition for occurrence
$2A_1$	$W(B_6)$	$\lvert A_1(q)\rvert^2$	$(q-1)^2(q^2+1)^2$	
			$(q-1)(q+1)^5$	
			$(q+1)^2(q^4-1)$	
			$(q+1)^2(q^4-1)$	
			$(q+1)(q^2+1)(q^3-1)$	
			q^6-1	
			$(q-1)(q+1)^2(q^3+1)$	
			$(q^2-1)(q^2+1)^2$	
			$(q^2-1)(q^4+1)$	
			$(q-1)(q^2+1)(q^3+1)$	
			$(q+1)^6$	
			$(q+1)^2(q^2+1)^2$	
			$(q+1)^3(q^3+1)$	
			$(q^3+1)^2$	
			$(q^2+1)(q^4+1)$	
			$(q+1)(q^5+1)$	
		$\lvert A_1(q^2)\rvert$	$(q+1)(q-1)^5$	
			$(q+1)^2(q-1)^4$	
			$(q^2+1)(q-1)^4$	
			$(q^2-1)^3$	
			$(q^2-1)^3$	
			$(q-1)(q^2-1)(q^3-1)$	
			$(q-1)^2(q^4-1)$	
			$(q^2-1)^3$	
			$(q-1)^2(q+1)^4$	
			$(q^2-1)(q^4-1)$	
			$(q^2-1)(q^4-1)$	

TABLE 6 (continued)

Type of Δ_J	Ω_J	$\lvert(M^g)_\sigma\rvert$	$\lvert(S^g)_\sigma\rvert$	condition for occurrence
$2A_1$	$W(B_6)$	$\lvert A_1(q^2)\rvert$	$(q+1)(q^2-1)(q^3-1)$	
			$(q-1)(q^2+1)(q^3-1)$	
			$(q^2-1)(q^4-1)$	
			$(q-1)(q^2-1)(q^3+1)$	
			$(q-1)^2(q^4+1)$	
			$(q^2-1)(q+1)^4$	
			$(q^3-1)(q+1)^3$	
			$(q+1)^2(q^4-1)$	
			$(q^2-1)(q^2+1)^2$	
			$(q+1)(q^2-1)(q^3+1)$	
			q^6-1	
			$(q^2+1)(q^4-1)$	
			$(q+1)(q^5-1)$	
			$(q^2-1)(q^4+1)$	
			$(q-1)(q^5+1)$	
			$(q^2+1)(q+1)^4$	
			$(q^2+1)^3$	
			$(q+1)(q^2+1)(q^3+1)$	
			$(q+1)^2(q^4+1)$	
			q^6+1	
A_2	$W(E_6) \times Z_2$	$\lvert A_2(q)\rvert$	$(q-1)^6$	
			$(q+1)(q-1)^5$	
			$(q+1)^2(q-1)^4$	
			$(q^3-1)(q-1)^3$	
			$(q^2-1)^3$	
			$(q-1)(q^2-1)(q^3-1)$	

TABLE 6 (continued)

Type of Δ_J	Ω_J	$\|(M^g)_\sigma\|$	$\|(S^g)_\sigma\|$	condition for occurrence
A_2	$W(E_6)\times Z_2$	$\|A_2(q)\|$	$(q-1)^2(q^4-1)$	
			$(q+1)^2(q^2-1)^2$	
			$(q+1)(q^2-1)(q^3-1)$	
			$(q^3-1)^2$	
			$(q^2-1)(q^4-1)$	
			$(q-1)(q^5-1)$	
			$(q-1)(q^2-1)(q^3+1)$	
			$(q-1)^2(q^2+1)^2$	
			$(q+1)(q^2+q+1)(q^3-1)$	
			$(q+1)^2(q^4-1)$	
			$(q+1)(q^5-1)$	
			q^6-1	
			$(q^2-1)(q^4+1)$	
			$(q-1)(q^2+1)(q^3+1)$	
			$(q^2+q+1)^3$	
			$(q+1)(q^2+q+1)(q^3+1)$	
			$(q^2+q+1)(q^4-q^2+1)$	
			q^6+q^3+1	
			$(q^2+q+1)(q^2-q+1)^2$	
		$\|{}^2A_2(q^2)\|$	$(q+1)^6$	
			$(q^2-1)(q+1)^4$	
			$(q^2-1)^2(q+1)^2$	
			$(q^3+1)(q+1)^3$	
			$(q^2-1)^3$	
			$(q+1)(q^2-1)(q^3+1)$	
			$(q+1)^2(q^4-1)$	
			$(q-1)^2(q^2-1)^2$	

TABLE 6 (continued)

Type of Δ_J	Ω_J	$\lvert (M^g)_\sigma \rvert$	$\lvert (S^g)_\sigma \rvert$	condition for occurrence
A_2	$W(E_6) \times Z_2$	$\lvert {}^2A_2(q^2) \rvert$	$(q-1)(q^2-1)(q^3+1)$	
			$(q^3+1)^2$	
			$(q^2-1)(q^4-1)$	
			$(q+1)(q^5+1)$	
			$(q+1)(q^2-1)(q^3-1)$	
			$(q+1)^2(q^2+1)^2$	
			$(q^2-1)(q^2-q+1)^2$	
			$(q+1)^2(q^4-1)$	
			$(q-1)(q^5+1)$	
			q^6-1	
			$(q^2-1)(q^4+1)$	
			$(q^2+q+1)(q^4-1)$	
			$(q^2-q+1)^3$	
			$(q-1)(q^2-q+1)(q^3-1)$	
			$(q^2-q+1)(q^4-q^2+1)$	
			q^6-q^3+1	
			$(q^2+q+1)(q^4+q^2+1)$	
$3A_1$	$W(F_4) \times Z_2$	$\lvert A_1(q) \rvert^3$	$(q-1)^5$	
			$(q+1)(q-1)^4$	
			$(q-1)(q^2-1)^2$	
			$(q-1)^2(q^3-1)$	
			$(q-1)(q^4-1)$	
			$(q^2-1)(q+1)^3$	
			$(q+1)(q^2-1)^2$	
			$(q^2-1)(q^3+1)$	
			$(q-1)(q^2+1)^2$	
			$(q+1)(q-1)^4$	

TABLE 6 (continued)

Type of Δ_J	Ω_J	$\lvert(M^g)_\sigma\rvert$	$\lvert(S^g)_\sigma\rvert$	condition for occurrence
$3A_1$	$W(F_4) \times Z_2$	$\lvert A_1(q)\rvert^3$	$(q-1)(q^2-1)^2$	
			$(q+1)(q^2-1)^2$	
			$(q^2-1)(q^3-1)$	
			$(q+1)(q^4-1)$	
			$(q+1)^5$	
			$(q^2-1)(q+1)^3$	
			$(q+1)(q^3+1)$	
			$(q+1)(q^2+1)^2$	
		$\lvert A_1(q^2)\rvert\lvert A_1(q)\rvert$	$(q+1)(q-1)^4$	
			$(q^2+1)(q-1)^3$	
			$(q-1)(q^2-1)^2$	
			$(q+1)(q^2-1)^2$	
			$(q-1)^2(q^3+1)$	
			$(q^2-1)(q^3-1)$	
			$(q-1)(q^4-1)$	
			$(q-1)(q^4+1)$	
			$(q+1)(q^4-1)$	
			$(q-1)(q^2-1)^2$	
			$(q-1)(q^4-1)$	
			$(q+1)(q^2-1)^2$	
			$(q^2-1)(q+1)^3$	
			$(q^2-1)(q^3+1)$	
			$(q+1)^2(q^3-1)$	
			$(q+1)(q^4-1)$	
			$(q+1)(q^4+1)$	
			$(q^2+1)(q+1)^3$	

TABLE 6 (continued)

Type of Δ_J	Ω_J	$\lvert(M^g)_\sigma\rvert$	$\lvert(S^g)_\sigma\rvert$	condition for occurrence
$3A_1$	$W(F_4) \times Z_2$	$\lvert A_1(q^3)\rvert$	$(q-1)^2(q^3-1)$	
			$(q-1)^2(q^3+1)$	
			$(q^2-1)(q^3-1)$	
			$(q^2-1)(q^3+1)$	
			$(q-1)(q^4-q^2+1)$	
			$(q-1)(q^2-q+1)^2$	
			$(q^2+q+1)(q^3-1)$	
			$(q^2-1)(q^3-1)$	
			$(q^2-1)(q^3+1)$	
			$(q+1)^2(q^3-1)$	
			$(q+1)^2(q^3+1)$	
			$(q+1)(q^4-q^2+1)$	
			$(q+1)(q^2-q+1)^2$	
			$(q+1)(q^2+q+1)^2$	
A_2+A_1	$S_6 \times Z_2$	$\lvert A_2(q)\rvert\lvert A_1(q)\rvert$	$(q-1)^5$	
			$(q+1)(q-1)^4$	
			$(q-1)(q^2-1)^2$	
			$(q-1)^2(q^3-1)$	
			$(q-1)^2(q+1)^3$	
			$(q^2-1)(q^3-1)$	
			$(q-1)(q^4-1)$	
			$(q-1)(q^2+q+1)^2$	
			$(q+1)(q^4-1)$	
			q^5-1	
			$(q^2+q+1)(q^3+1)$	
		$\lvert {}^2A_2(q^2)\rvert\lvert A_1(q)\rvert$	$(q+1)^5$	
			$(q^2-1)(q+1)^3$	

TABLE 6 (continued)

Type of Δ_J	Ω_J	$\lvert(M^g)_\sigma\rvert$	$\lvert(S^g)_\sigma\rvert$	condition for occurrence
A_2+A_1	$S_6 \times Z_2$	$\lvert{}^2A_2(q^2)\rvert\,\lvert A_1(q)\rvert$	$(q+1)(q^2-1)^2$	
			$(q+1)^2(q^3+1)$	
			$(q-1)(q^2-1)^2$	
			$(q^2-1)(q^3+1)$	
			$(q+1)(q^4-1)$	
			$(q^2-q+1)(q^3+1)$	
			$(q-1)(q^4-1)$	
			q^5+1	
			$(q-1)(q^4+q^2+1)$	
A_3	$W(B_5)$	$\lvert A_3(q)\rvert$	$(q-1)^5$	
			$(q+1)(q-1)^4$	
			$(q+1)^2(q-1)^3$	
			$(q+1)^2(q-1)^3$	
			$(q-1)^2(q^3-1)$	
			$(q-1)^2(q+1)^3$	
			$(q^2-1)(q^3-1)$	
			$(q-1)(q^4-1)$	
			$(q-1)(q^4-1)$	
			$(q-1)(q+1)^4$	
			$(q+1)^2(q^3-1)$	
			$(q+1)(q^4-1)$	
			$(q-1)(q^2+1)^2$	
			$(q^2-1)(q^3+1)$	
			(q^5-1)	
			$(q^2+1)(q^3+1)$	
			$(q^2+1(q+1)^3$	
			$(q+1)(q^4+1)$	

TABLE 6 (continued)

Type of Δ_J	Ω_J	$\lvert(M^g)_\sigma\rvert$	$\lvert(S^g)_\sigma\rvert$	condition for occurrence
A_3	$W(B_5)$	$\lvert{}^2A_3(q^2)\rvert$	$(q^2-1)(q-1)^3$	
			$(q-1)(q^2-1)^2$	
			$(q^2+1)(q-1)^3$	
			$(q+1)(q^2-1)^2$	
			$(q+1)(q^2-1)^2$	
			$(q^2-1)(q^3-1)$	
			$(q-1)(q^4-1)$	
			$(q-1)^2(q^3+1)$	
			$(q^2-1)(q+1)^3$	
			$(q+1)(q^4-1)$	
			$(q^2+1)(q^3-1)$	
			$(q+1)(q^4-1)$	
			$(q^2-1)(q^3+1)$	
			$(q-1)(q^4+1)$	
			$(q+1)^2(q^3+1)$	
			$(q+1)^5$	
			$(q+1)(q^2+1)^2$	
			q^5+1	
$[4A_1]'$	$W(B_4)$	$\lvert A_1(q)\rvert^4$	$(q-1)^4$	
			$(q+1)(q-1)^3$	
			$(q^2-1)^2$	
			$(q-1)(q+1)^3$	
			$(q+1)^4$	
		$\lvert A_1(q^2)\rvert\lvert A_1(q)\rvert^2$	$(q-1)^2(q^2-1)$	
			$(q-1)^2(q^2+1)$	
			$(q^2-1)^2$	
			q^4-1	

TABLE 6 (continued)

Type of Δ_J	Ω_J	$\lvert(M^g)_\sigma\rvert$	$\lvert(S^g)_\sigma\rvert$	condition for occurrence
$[4A_1]'$	$W(B_4)$	$\lvert A_1(q^2)\rvert\lvert A_1(q)\rvert^2$	$(q+1)^2(q^2-1)$	
			$(q+1)^2(q^2+1)$	
		$\lvert A_1(q^2)\rvert^2$	$(q^2-1)^2$	
			q^4-1	
			$(q^2+1)^2$	
		$\lvert A_1(q^3)\rvert\lvert A_1(q)\rvert$	$(q-1)(q^3-1)$	
			$(q-1)(q^3+1)$	
			$(q+1)(q^3-1)$	
			$(q+1)(q^3+1)$	
		$\lvert A_1(q^4)\rvert$	q^4-1	
			q^4+1	
$[4A_1]''$	H_2	$\lvert A_1(q)\rvert^4$	$(q-1)^4$	
			$(q-1)^2(q^2-1)$	
			$(q^2-1)^2$	
			$(q-1)(q^3-1)$	
			q^4-1	
			$(q+1)^4$	
			$(q+1)^2(q^2-1)$	
			$(q+1)(q^3+1)$	
			$(q^2+1)^2$	
		$\lvert A_1(q^2)\rvert\lvert A_1(q)\rvert^2$	$(q-1)^2(q^2-1)$	
			$(q-1)^2(q^2+1)$	
			$(q^2-1)^2$	
			$(q+1)^2(q^2-1)$	
			$(q+1)(q^3+1)$	
			$(q+1)(q^3-1)$	
			q^4-1	
			q^4+1	

TABLE 6 (continued)

Type of Δ_J	Ω_J	$\lvert(M^g)_\sigma\rvert$	$\lvert(S^g)_\sigma\rvert$	condition for occurrence
$[4A_1]''$	H_2	$\lvert A_1(q^2)\rvert\lvert A_1(q)\rvert^2$	$(q+1)^2(q^2+1)$	
		$\lvert A_1(q^3)\rvert\lvert A_1(q)\rvert$	$(q-1)(q^3-1)$	
			$(q-1)(q^3+1)$	
			$(q+1)(q^3-1)$	
			$(q+1)(q^3+1)$	
			(q^4-q^2+1)	
			$(q^2-q+1)^2$	
			$(q^2+q+1)^2$	
		$\lvert A_1(q^2)\rvert^2$	$(q-1)^4$	
			$(q-1)^2(q^2-1)$	
			$(q^2-1)^2$	
			$(q^2-1)^2$	
			$(q-1)(q^3-1)$	
			q^4-1	
			q^4-1	
			$(q+1)^4$	
			$(q+1)^2(q^2-1)$	
			$(q+1)(q^3+1)$	
			$(q^2+1)^2$	
		$\lvert A_1(q^4)\rvert$	$(q-1)^2(q^2-1)$	
			$(q-1)^2(q^2+1)$	
			$(q^2-1)^2$	
			$(q+1)^2(q^2-1)$	
			$(q-1)(q^3+1)$	
			$(q+1)(q^3-1)$	
			q^4-1	
			q^4+1	
			$(q+1)^2(q^2+1)$	

TABLE 6 (continued)

Type of Δ_J	Ω_J	$\lvert(M^g)_\sigma\rvert$	$\lvert(S^g)_\sigma\rvert$	condition for occurrence
A_2+2A_1	$S_4 \times Z_2 \times Z_2$	$\lvert A_2(q)\rvert\lvert A_1(q)\rvert^2$	$(q-1)^4$	
			$(q-1)^2(q^2-1)$	
			$(q^2-1)^2$	
			$(q-1)(q^3-1)$	
			q^4-1	
		$\lvert A_2(q)\rvert\lvert A_1(q^2)\rvert$	$(q+1)^2(q^2-1)$	
			$(q^2-1)^2$	
			$(q-1)^2(q^2-1)$	
			$(q-1)(q^3+1)$	
			$(q-1)^2(q^2+1)$	
		$\lvert{}^2A_2(q^2)\rvert\lvert A_1(q)\rvert^2$	$(q+1)^4$	
			$(q+1)^2(q^2-1)$	
			$(q^2-1)^2$	
			$(q+1)(q^3+1)$	
			q^4-1	
		$\lvert{}^2A_2(q^2)\rvert\lvert A_1(q^2)\rvert$	$(q-1)^2(q^2-1)$	
			$(q^2-1)^2$	
			$(q+1)^2(q^2-1)$	
			$(q+1)(q^3-1)$	
			$(q+1)^2(q^2+1)$	
$2A_2$	$(S_3 \times Z_2) \wr Z_2$	$\lvert A_2(q)\rvert^2$	$(q-1)^4$	
			$(q-1)^2(q^2-1)$	
			$(q-1)(q^3-1)$	
			$(q^2-1)^2$	
			$(q+1)(q^3-1)$	
			$(q^2+q+1)^2$	

TABLE 6 (continued)

Type of Δ_J	Ω_J	$\vert(M^g)_\sigma\vert$	$\vert(S^g)_\sigma\vert$	condition for occurrence
$2A_2$	$(S_3 \times Z_2) \wr Z_2$	$\vert A_2(q^2)\vert$	$(q^2-1)^2$	
			$(q+1)^2(q^2-1)$	
			$(q+1)^2(q^2+q+1)$	
			$(q-1)^2(q^2-1)$	
			$(q-1)^2(q^2-q+1)$	
			$(q^2-1)^2$	
			$(q+1)(q^3-1)$	
			$(q-1)(q^3+1)$	
			q^4+q+1	
		$\vert {}^2A_2(q^2)\vert^2$	$(q+1)^4$	
			$(q+1)^2(q^2-1)$	
			$(q+1)(q^3+1)$	
			$(q^2-1)^2$	
			$(q-1)(q^3+1)$	
			$(q^2-q+1)^2$	
		$\vert {}^2A_2(q^2)\vert\,\vert A_2(q)\vert$	$(q^2-1)^2$	
			q^4-1	
			q^4+q^2+1	
			q^4-1	
		$\vert {}^2A_2(q^4)\vert$	$(q^2+1)^2$	
			q^4-1	
			q^4-q^2+1	
			q^4-1	
A_3+A_1	$S_4 \times Z_2 \times Z_2$	$\vert A_3(q)\vert\,\vert A_1(q)\vert$	$(q-1)^4$	
			$(q-1)^2(q^2-1)$	
			$(q^2-1)^2$	
			$(q-1)(q^3-1)$	

TABLE 6 (continued)

Type of Δ_J	Ω_J	$\|(M^g)_\sigma\|$	$\|(S^g)_\sigma\|$	condition for occurrence
A_3+A_1	$S_4 \times Z_2 \times Z_2$	$\|A_3(q)\|\|A_1(q)\|$	q^4-1	
			$(q-1)^2(q^2-1)$	
			$(q^2-1)^2$	
			$(q+1)^2(q^2-1)$	
			$(q+1)(q^3-1)$	
			$(q+1)^2(q^2+1)$	
		$\|{}^2A_3(q^2)\|\|A_1(q)\|$	$(q+1)^2(q^2-1)$	
			$(q^2-1)^2$	
			$(q-1)^2(q^2-1)$	
			$(q-1)(q^3+1)$	
			$(q-1)(q^3-q^2+q-1)$	
			$(q+1)^4$	
			$(q+1)^2(q^2-1)$	
			$(q^2-1)^2$	
			$(q+1)(q^3+1)$	
			q^4-1	
A_4	$S_5 \times Z_2$	$\|A_4(q)\|$	$(q-1)^4$	
			$(q^2-1)(q-1)^2$	
			$(q^2-1)^2$	
			$(q-1)(q^3-1)$	
			$(q+1)(q^3-1)$	
			q^4-1	
			$q^4+q^3+q^2+q+1$	
		$\|{}^2A_4(q^2)\|$	$(q+1)^4$	
			$(q+1)^2(q^2-1)$	
			$(q^2-1)^2$	
			$(q+1)(q^3+1)$	
			$(q-1)(q^3+1)$	

TABLE 6 (continued)

Type of Δ_J	Ω_J	$\|(M^g)_\sigma\|$	$\|(S^g)_\sigma\|$	condition for occurrence
A_4	$S_5 \times Z_2$	$\|{}^2A_4(q^2)\|$	q^4-1	
			$q^4-q^3+q^2-q+1$	
D_4	$W(F_4)$	$\|D_4(q)\|$	$(q-1)^4$	
			$(q-1)^2(q^2-1)$	
			$(q^2-1)^2$	
			$(q-1)(q^3-1)$	
			q^4-1	
			$(q+1)^4$	
			$(q+1)^2(q^2-1)$	
			$(q+1)(q^3+1)$	
			$(q^2+1)^2$	
		$\|{}^2D_4(q^2)\|$	$(q^2-1)(q-1)^2$	
			$(q^2+1)(q-1)^2$	
			$(q^2-1)^2$	
			$(q^2-1)(q+1)^2$	
			$(q-1)(q^3+1)$	
			$(q+1)(q^3+1)$	
			q^4-1	
			q^4+1	
			$(q^2+1)(q+1)^2$	
		$\|{}^3D_4(q^3)\|$	$(q-1)(q^3-1)$	
			$(q-1)(q^3+1)$	
			$(q+1)(q^3-1)$	
			$(q+1)(q^3+1)$	
			q^4-q^2+1	
			$(q^2-q+1)^2$	
			$(q^2+q+1)^2$	

TABLE 6 (continued)

Type of Δ_J	Ω_J	$\lvert (M^g)_\sigma \rvert$	$\lvert (S^g)_\sigma \rvert$	condition for occurrence
$5A_1$	H_1	$\lvert A_1(q)\rvert^5$	$(q-1)^3$	$2\lvert q-1$
			$(q-1)(q^2-1)$	$2\lvert q-1$
			$(q+1)(q^2-1)$	$2\lvert q-1$
			$(q+1)^3$	$2\lvert q-1$
		$\lvert A_1(q^2)\rvert\lvert A_1(q)\rvert^3$	$(q-1)(q^2-1)$	$2\lvert q-1$
			$(q+1)(q^2-1)$	$2\lvert q-1$
			$(q-1)(q^2+1)$	$2\lvert q-1$
			$(q+1)(q^2+1)$	$2\lvert q-1$
		$\lvert A_1(q^3)\rvert\lvert A_1(q)\rvert^2$	q^3-1	$2\lvert q-1$
			q^3+1	$2\lvert q-1$
		$\lvert A_1(q^4)\rvert\lvert A_1(q)\rvert$	$(q-1)(q^2-1)$	$2\lvert q-1$
			$(q-1)(q^2+1)$	$2\lvert q-1$
			$(q+1)(q^2-1)$	$2\lvert q-1$
			$(q+1)(q^2+1)$	$2\lvert q-1$
		$\lvert A_1(q^2)\rvert^2\lvert A_1(q)\rvert$	$(q-1)^3$	$2\lvert q-1$
			$(q-1)(q^2-1)$	$2\lvert q-1$
			$(q-1)(q^2-1)$	$2\lvert q-1$
			$(q+1)(q^2-1)$	$2\lvert q-1$
			$(q+1)(q^2-1)$	$2\lvert q-1$
			$(q+1)^3$	$2\lvert q-1$
A_2+3A_1	$S_3\times Z_2\times Z_2$	$\lvert A_2(q)\rvert\lvert A_1(q)\rvert^3$	$(q-1)^3$	
			$(q-1)(q^2-1)$	
		$\lvert A_2(q)\rvert\lvert A_1(q^2)\rvert\lvert A_1(q)\rvert$	$(q-1)(q^2-1)$	
			$(q+1)(q^2-1)$	
		$\lvert A_2(q)\rvert\lvert A_1(q^3)\rvert$	q^3-1	
			$(q+1)(q^2+q+1)$	

TABLE 6 (continued)

Type Δ_J	Ω_J	$\|(M^g)_\sigma\|$	$\|(S^g)_\sigma\|$	condition for occurrence
A_2+3A_1	$S_3\times Z_2\times Z_2$	$\|{}^2A_2(q^2)\|\|A_1(q)\|^3$	$(q+1)(q^2-1)$	
			$(q+1)^3$	
		$\|{}^2A_2(q^2)\|\|A_1(q^2)\|\|A_1(q)\|$	$(q-1)(q^2-1)$	
			$(q+1)(q^2-1)$	
		$\|{}^2A_2(q^2)\|\|A_1(q^3)\|$	q^3+1	
			$(q-1)(q^2-q+1)$	
$2A_2+A_1$	$S_3\times Z_2\times Z_2$	$\|A_2(q)\|^2\|(A_1(q)\|$	$(q-1)^3$	
			$(q-1)(q^2-1)$	
			q^3-1	
		$\|A_2(q^2)\|\|A_1(q)\|$	$(q-1)(q^2-1)$	
			$(q+1)(q^2-1)$	
			$(q-1)(q^2-q+1)$	
		$\|{}^2A_2(q^2)\|^2\|A_1(q)\|$	$(q+1)^3$	
			$(q+1)(q^2-1)$	
			q^3+1	
		$\|A_2(q^2)\|\|A_1(q)\|$	$(q-1)(q^2-1)$	
			$(q+1)(q^2-1)$	
			$(q+1)(q^2+q+1)$	
$[A_3+2A_1]'$	$D_8\times Z_2$	$\|A_3(q)\|\|A_1(q)\|^2$	$(q-1)^3$	
			$(q-1)(q^2-1)$	
			$(q+1)(q^2-1)$	
		$\|{}^2A_3(q^2)\|\|A_1(q)\|^2$	$(q-1)(q^2-1)$	
			$(q+1)(q^2-1)$	
			$(q+1)^3$	
		$\|A_3(q)\|\|A_1(q^2)\|$	$(q+1)(q^2-1)$	
			$(q-1)(q^2+1)$	
		$\|{}^2A_3(q^2)\|\|A_1(q^2)\|$	$(q+1)(q^2-1)$	
			$(q+1)(q^2+1)$	

Type of Δ_J	Ω_J	$\|(M^g)_\sigma\|$	$\|(S^g)_\sigma\|$	condition for occurrence
$[A_3+2A_1]''$	$S_4\times Z_2\times Z_2$	$\|A_3(q)\|\|A_1(q)\|^2$	$(q-1)^3$	
			$(q-1)(q^2-1)$	
			$(q+1)(q^2-1)$	
			q^3-1	
			$(q+1)(q^2+1)$	
		$\|{}^2A_3(q^2)\|\|A_1(q)\|^2$	$(q+1)^3$	
			$(q+1)(q^2-1)$	
			$(q-1)(q^2-1)$	
			q^3+1	
			$(q-1)(q^2+1)$	
		$\|A_3(q)\|\|A_1(q^2)\|$	$(q+1)^3$	
			$(q+1)(q^2-1)$	
			$(q-1)(q^2-1)$	
			q^3+1	
			$(q-1)(q^2+1)$	
		$\|{}^2A_3(q^2)\|\|A_1(q^2)\|$	$(q-1)^3$	
			$(q-1)(q^2-1)$	
			$(q+1)(q^2-1)$	
			q^3-1	
			$(q+1)(q^2+1)$	
A_3+A_2	$D_8\times Z_2$	$\|A_3(q)\|\|A_2(q)\|$	$(q-1)^3$	
			$(q-1)(q^2-1)$	
			$(q+1)(q^2-1)$	
		$\|{}^2A_3(q^2)\|\|A_2(q)\|$	$(q-1)(q^2-1)$	
			$(q-1)(q^2+1)$	
		$\|A_3(q)\|\|{}^2A_2(q^2)\|$	$(q+1)(q^2-1)$	
			$(q+1)(q^2+1)$	

TABLE 6 (continued)

Type of Δ_J	Ω_J	$\lvert(M^g)_\sigma\rvert$	$\lvert(S^g)_\sigma\rvert$	condition for occurrence
A_3+A_2	$D_8 \times Z_2$	$\lvert{}^2A_3(q^2)\rvert\lvert{}^2A_2(q^2)\rvert$	$(q-1)(q^2-1)$	
			$(q+1)(q^2-1)$	
			$(q+1)^3$	
A_4+A_1	$S_3 \times Z_2$	$\lvert A_4(q)\rvert\lvert A_1(q)\rvert$	$(q-1)^3$	
			$(q-1)(q^2-1)$	
		$\lvert{}^2A_4(q^2)\rvert\lvert A_1(q)\rvert$	q^3-1	
			$(q+1)^3$	
			$(q+1)(q^2-1)$	
			q^3+1	
A_5	$S_3 \times Z_2 \times Z_2$	$\lvert A_5(q)\rvert$	$(q-1)^3$	
			$(q-1)(q^2-1)$	
			q^3-1	
			$(q-1)(q^2-1)$	
			$(q+1)(q^2-1)$	
			$(q+1)(q^2+q+1)$	
		$\lvert{}^2A_5(q^2)\rvert$	$(q+1)(q^2-1)$	
			$(q-1)(q^2-1)$	
			$(q-1)(q^2-q+1)$	
			$(q+1)^3$	
			$(q+1)(q^2-1)$	
			q^3+1	
D_4+A_1	$W(B_3)$	$\lvert D_4(q)\rvert\lvert A_1(q)\rvert$	$(q-1)^3$	
			$(q-1)(q^2-1)$	
			$(q+1)(q^2-1)$	
			$(q+1)^3$	
		$\lvert{}^2D_4(q^2)\rvert\lvert A_1(q)\rvert$	$(q-1)(q^2-1)$	
			$(q-1)(q^2+1)$	

TABLE 6 (continued)

Type of Δ_J	Ω_J	$\lvert(M^g)_\sigma\rvert$	$\lvert(S^g)_\sigma\rvert$	condition for occurrence
D_4+A_1	$W(B_3)$	$\lvert{}^2D_4(q^2)\rvert\lvert A_1(q)\rvert$	$(q+1)(q^2-1)$	
			$(q+1)(q^2+1)$	
		$\lvert{}^3D_4(q^3)\rvert\lvert A_1(q)\rvert$	q^3-1	
			q^3+1	
D_5	$S_4\times Z_2$	$\lvert D_5(q)\rvert$	$(q-1)^3$	
			$(q-1)(q^2-1)$	
			$(q+1)(q^2-1)$	
			q^3-1	
			$(q+1)(q^2+1)$	
		$\lvert{}^2D_5(q^2)\rvert$	$(q+1)^3$	
			$(q+1)(q^2-1)$	
			$(q-1)(q^2-1)$	
			q^3+1	
			$(q-1)(q^2+1)$	
$A_2'+4A_1$	$S_4\times Z_2$	$\lvert A_2(q)\rvert\lvert A_1(q)\rvert^4$	$(q-1)^2$	$2\mid q-1$
		$\lvert A_2(q)\rvert\lvert A_1(q^2)\rvert\lvert A_1(q)\rvert^2$	q^2-1	$2\mid q-1$
		$\lvert A_2(q)\rvert\lvert A_1(q^3)\rvert\lvert A_1(q)\rvert$	q^2+q+1	$2\mid q-1$
		$\lvert A_2(q)\rvert\lvert A_1(q^2)\rvert^2$	$(q-1)^2$	$2\mid q-1$
		$\lvert A_2(q)\rvert\lvert A_1(q^4)\rvert$	q^2-1	$2\mid q-1$
		$\lvert{}^2A_2(q^2)\rvert\lvert A_1(q)\rvert^4$	$(q+1)^2$	$2\mid q-1$
		$\lvert{}^2A_2(q^2)\rvert\lvert A_1(q^2)\rvert\lvert A_1(q)\rvert^2$	q^2-1	$2\mid q-1$
		$\lvert{}^2A_2(q^2)\rvert\lvert A_1(q^3)\rvert\lvert A_1(q)\rvert$	q^2-q+1	$2\mid q-1$
		$\lvert{}^2A_2(q^2)\rvert\lvert A_1(q^2)\rvert^2$	$(q+1)^2$	$2\mid q-1$
		$\lvert{}^2A_2(q^2)\rvert\lvert A_1(q^4)\rvert$	q^2+q+1	$2\mid q-1$
$2A_2+2A_1$	D_8	$\lvert A_2(q)\rvert^2\lvert A_1(q)\rvert^2$	$(q-1)^2$	
		$\lvert A_2(q)\rvert\lvert{}^2A_2(q^2)\rvert\lvert A_1(q^2)\rvert$	q^2-1	

TABLE 6 (continued)

Type of Δ_J	Ω_J	$\lvert(M^g)_\sigma\rvert$	$\lvert(S^g)_\sigma\rvert$	condition for occurrence
$2A_2+2A_1$	D_8	$\lvert A_2(q^2)\rvert\lvert A_1(q)\rvert^2$	q^2-1	
		$\lvert {}^2A_2(q^4)\rvert\lvert A_1(q^2)\rvert$	q^2+1	
		$\lvert {}^2A_2(q^2)\rvert^2\lvert A_1(q)\rvert^2$	$(q+1)^2$	
$3A_2$	$S_3\times S_3\times Z_2$	$\lvert A_2(q)\rvert^3$	$(q-1)^2$	$3\mid q-1$
			(q^2-1)	$3\mid q-1$
			q^2+q+1	$3\mid q-1$
		$\lvert A_2(q^2)\rvert\lvert {}^2A_2(q^2)\rvert$	$(q-1)^2$	$3\mid q+1$
			q^2-1	$3\mid q+1$
			q^2+q+1	$3\mid q+1$
		$\lvert {}^2A_2(q^2)\rvert^3$	$(q+1)^2$	$3\mid q+1$
			q^2-1	$3\mid q+1$
			q^2-q+1	$3\mid q+1$
		$\lvert A_2(q^2)\rvert\lvert A_2(q)\rvert$	$(q+1)^2$	$3\mid q-1$
			q^2-1	$3\mid q-1$
			q^2-q+1	$3\mid q-1$
		$\lvert A_2(q^3)\rvert$	$(q-1)^2$	$3\mid q-1$
			q^2-1	$3\mid q-1$
			q^2+q+1	$3\mid q-1$
		$\lvert {}^2A_2(q^6)\rvert$	$(q+1)^2$	$3\mid q+1$
			q^2-1	$3\mid q+1$
			q^2-q+1	$3\mid q+1$
A_3+3A_1	$Z_2\times Z_2\times Z_2$	$\lvert A_3(q)\rvert\lvert A_1(q)\rvert^3$	$(q-1)^2$	$2\mid q-1$
			q^2-1	$2\mid q-1$
		$\lvert {}^2A_3(q^2)\rvert\lvert A_1(q)\rvert^3$	$(q+1)^2$	$2\mid q-1$
			(q^2-1)	$2\mid q-1$

TABLE 6 (continued)

Type of Δ_J	Ω_J	$\lvert(M^g)_\sigma\rvert$	$\lvert(S^g)_\sigma\rvert$	condition for occurrence
A_3+3A_1	$Z_2\times Z_2\times Z_2$	$\lvert A_3(q)\rvert\lvert A_1(q^2)\rvert\lvert A_1(q)\rvert$	$(q+1)^2$	$2\mid q-1$
			q^2-1	$2\mid q-1$
		$\lvert {}^2A_3(q^2)\rvert\lvert A_1(q^2)\rvert\lvert A_1(q)\rvert$	$(q-1)^2$	$2\mid q-1$
			q^2-1	$2\mid q-1$
$A_3+A_2+A_1$	$Z_2\times Z_2$	$\lvert A_3(q)\rvert\lvert A_2(q)\rvert\lvert A_1(q)\rvert$	$(q-1)^2$	
			q^2-1	
		$\lvert {}^2A_3(q^2)\rvert\lvert {}^2A_2(q^2)\rvert\lvert A_1(q)\rvert$	$(q+1)^2$	
			q^2-1	
$[2A_3]'$	D_8	$\lvert A_3(q)\rvert^2$	$(q-1)^2$	
		$\lvert {}^2A_3(q^2)\rvert\lvert A_3(q)\rvert$	q^2-1	
		$\lvert A_3(q^2)\rvert$	q^2-1	
		$\lvert {}^2A_3(q^2)\rvert^2$	q^2-1	
		$\lvert {}^2A_3(q^4)\rvert$	$(q+1)^2$	
$[2A_3]''$	$(Z_2\times Z_2)\wr Z_2$	$\lvert A_3(q)\rvert^2$	$(q-1)^2$	
			q^2-1	
			$(q+1)^2$	
		$\lvert {}^2A_3(q^2)\rvert\lvert A_3(q)\rvert$	q^2-1	
			q^2+1	
		$\lvert A_3(q^2)\rvert$	$(q-1)^2$	
			q^2-1	
			q^2-1	
			$(q+1)^2$	
		$\lvert {}^2A_3(q^2)\rvert^2$	$(q-1)^2$	
			q^2-1	
			$(q+1)^2$	
		$\lvert {}^2A_3(q^4)\rvert$	q^2-1	
			q^2+1	

TABLE 6 (continued)

Type of Δ_J	Ω_J	$\lvert(M^g)_\sigma\rvert$	$\lvert(S^g)_\sigma\rvert$	condition for occurrence
A_4+2A_1	$Z_2\times Z_2$	$\lvert A_4(q)\rvert\lvert A_1(q)\rvert^2$	$(q-1)^2$	
		$\lvert {}^2A_4(q^2)\rvert\lvert A_1(q^2)\rvert$	q^2-1	
		$\lvert {}^2A_4(q^2)\rvert\lvert A_1(q)\rvert^2$	$(q+1)^2$	
		$\lvert A_4(q)\rvert\lvert A_1(q^2)\rvert$	q^2-1	
A_4+A_2	$Z_2\times Z_2$	$\lvert A_4(q)\rvert\lvert A_2(q)\rvert$	$(q-1)^2$	
			q^2-1	
		$\lvert {}^2A_4(q^2)\rvert\lvert {}^2A_2(q^2)\rvert$	$(q+1)^2$	
			q^2-1	
$[A_5+A_1]'$	$Z_2\times Z_2$	$\lvert A_5(q)\rvert\lvert A_1(q)\rvert$	$(q-1)^2$	
			q^2-1	
		$\lvert {}^2A_5(q^2)\rvert\lvert A_1(q)\rvert$	q^2-1	
			$(q+1)^2$	
$[A_5+A_1]''$	$S_3\times Z_2$	$\lvert A_5(q)\rvert\lvert A_1(q)\rvert$	$(q-1)^2$	
			q^2-1	
			q^2+q+1	
		$\lvert {}^2A_5(q^2)\rvert\lvert A_1(q)\rvert$	$(q+1)^2$	
			q^2-1	
			q^2-q+1	
A_6	$Z_2\times Z_2$	$\lvert A_6(q)\rvert$	$(q-1)^2$	
			q^2-1	
		$\lvert {}^2A_6(q^2)\rvert$	$(q+1)^2$	
			$q^2-1)$	
D_4+2A_1	$D_8\times Z_2$	$\lvert D_4(q)\rvert\lvert A_1(q)\rvert^2$	$(q-1)^2$	$2\mid q-1$
			q^2-1	$2\mid q-1$
			q^2+q+1	$2\mid q-1$
		$\lvert D_4(q)\rvert\lvert A_1(q^2)\rvert$	q^2-1	$2\mid q-1$
			q^2+1	$2\mid q-1$
		$\lvert {}^2D_4(q^2)\rvert\lvert A_1(q)\rvert^2$	q^2-1	$2\mid q-1$
			q^2+1	$2\mid q-1$

TABLE 6 (continued)

Type of Δ_J	Ω_J	$\|(M^g)_\sigma\|$	$\|(S^g)_\sigma\|$	condition for occurrence
D_4+2A_1	$D_8 \times Z_2$	$\|{}^2D_4(q^2)\|\|A_1(q^2)\|$	$(q+1)^2$	$2\|q-1$
			$(q^2-1$	$2\|q-1$
			$(q-1)^2$	$2\|q-1$
D_4+A_2	$S_3 \times Z_2$	$\|D_4(q)\|\|A_2(q)\|$	$(q-1)^2$	
		$\|{}^2D_4(q^2)\|\|{}^2A_2(q^2)\|$	(q^2-1)	
		$\|{}^3D_4(q^3)\|\|A_2(q)\|$	q^2+q+1	
		$\|D_4(q)\|\|{}^2A_2(q^2)\|$	$(q+1)^2$	
		$\|{}^2D_4(q^2)\|\|A_2(q)\|$	q^2-1	
		$\|{}^3D_4(q^3)\|\|{}^2A_2(q^2)\|$	q^2-q+1	
D_5+A_1	$Z_2 \times Z_2$	$\|D_5(q)\|\|A_1(q)\|$	$(q-1)^2$	
			q^2-1	
		$\|{}^2D_5(q^2)\|\|A_1(q)\|$	q^2-1	
			$(q+1)^2$	
D_6	D_8	$\|D_6(q)\|$	$(q-1)^2$	
			q^2-1	
			$(q+1)^2$	
		$\|{}^2D_6(q^2)\|$	q^2-1	
			q^2+1	
E_6	$S_3 \times Z_2$	$\|E_6(q)\|$	$(q-1)^2$	
			q^2-1	
			q^2+q+1	
		$\|{}^2E_6(q^2)\|$	$(q+1)^2$	
			q^2-1	
			q^2-q+1	
$3A_2+A_1$	$S_3 \times Z_2$	$\|A_2(q)\|^3\|A_1(q)\|$	$q-1$	$3\|q-1$
		$\|A_2(q^2)\|\|{}^2A_2(q^2)\|\|A_1(q)\|$	$q-1$	$3\|q+1$
		$\|{}^2A_2(q^2)\|^3\|A_1(q)\|$	$q+1$	$3\|q+1$
		$\|A_2(q^2)\|\|A_2(q)\|\|A_1(q)\|$	$q+1$	$3\|q-1$

TABLE 6 (continued)

Type of Δ_J	Ω_J	$\lvert(M^g)_\sigma\rvert$	$\lvert(S^g)_\sigma\rvert$	condition for occurrence
$3A_2+A_1$	$S_3 \times Z_2$	$\lvert A_2(q^3)\rvert\lvert A_1(q)\rvert$	q-1	3|q-1
		$\lvert {}^2A_2(q^6)\rvert\lvert A_1(q)\rvert$	q+1	3|q+1
$A_3+A_2+2A_1$	$Z_2 \times Z_2$	$\lvert A_3(q)\rvert\lvert A_2(q)\rvert\lvert A_1(q)\rvert^2$	q-1	2|q-1
		$\lvert {}^2A_3(q^2)\rvert\lvert {}^2A_2(q^2)\rvert\lvert A_1(q)\rvert^2$	q+1	2|q-1
		$\lvert {}^2A_3(q^2)\rvert\lvert A_2(q)\rvert\lvert A_1(q^2)\rvert$	q-1	2|q-1
		$\lvert A_3(q)\rvert\lvert {}^2A_2(q^2)\rvert\lvert A_1(q^2)\rvert$	q+1	2|q-1
$2A_3+A_1$	$Z_2\times Z_2\times Z_2$	$\lvert A_3(q)\rvert^2\lvert A_1(q)\rvert$	q-1	4|q-1
			q+1	4|q-1
		$\lvert {}^2A_3(q^2)\rvert^2\lvert A_1(q)\rvert$	q-1	4|q+1
			q+1	4|q+1
		$\lvert A_3(q^2)\rvert\lvert A_1(q)\rvert$	q-1	4|q+1
			q+1	4|q+1
			q-1	4|q-1
			q+1	4|q-1
$A_4+A_2+A_1$	Z_2	$\lvert A_4(q)\rvert\lvert A_2(q)\rvert\lvert A_1(q)\rvert$	q-1	
		$\lvert {}^2A_4(q^2)\rvert\lvert {}^2A_2(q^2)\rvert\lvert A_1(q)\rvert$	q+1	
A_4+A_3	Z_2	$\lvert A_4(q)\rvert\lvert A_3(q)\rvert$	q-1	
		$\lvert {}^2A_4(q^2)\rvert\lvert {}^2A_3(q^2)\rvert$	q+1	
A_5+2A_1	Z_2	$\lvert A_5(q)\rvert\lvert A_1(q)\rvert^2$	q-1	2|q-1
		$\lvert {}^2A_5(q^2)\rvert\lvert A_1(q)\rvert^2$	q+1	2|q-1
A_5+A_2	$Z_2\times Z_2$	$\lvert A_5(q)\rvert\lvert A_2(q)\rvert$	q-1	3|q-1
			q+1	3|q-1
		$\lvert {}^2A_5(q^2)\rvert\lvert {}^2A_2(q^2)\rvert$	q-1	3|q+1
			q+1	3|q+1
A_6+A_1	Z_2	$\lvert A_6(q)\rvert\lvert A_1(q)\rvert$	q-1	
		$\lvert {}^2A_6(q^2)\rvert\lvert A_1(q)\rvert$	q+1	
$[A_7]'$	Z_2	$\lvert A_7(q)\rvert$	q-1	
		$\lvert {}^2A_7(q^2)\rvert$	q+1	
$[A_7]''$	$Z_2\times Z_2$	$\lvert A_7(q)\rvert$	q-1	2|q-1

TABLE 6 (continued)

Type of Δ_J	Ω_J	$\lvert(M^g)_\sigma\rvert$	$\lvert(S^g)_\sigma\rvert$	condition for occurrence
$[A_7]''$	$Z_2 \times Z_2$	$\lvert A_7(q)\rvert$	q+1	2|q-1
		$\lvert {}^2A_7(q^2)\rvert$	q-1	2|q-1
			q+1	2|q-1
D_4+A_3	$Z_2 \times Z_2$	$\lvert D_4(q)\rvert\lvert A_3(q)\rvert$	q-1	2|q-1
		$\lvert {}^2D_4(q^2)\rvert\lvert {}^2A_3(q^2)\rvert$	q-1	2|q-1
		$\lvert {}^2D_4(q^2)\rvert\lvert A_3(q)\rvert$	q+1	2|q-1
		$\lvert D_4(q)\rvert\lvert {}^2A_3(q^2)\rvert$	q+1	2|q-1
D_5+2A_1	$Z_2 \times Z_2$	$\lvert D_5(q)\rvert\lvert A_1(q)\rvert^2$	q-1	2|q-1
		$\lvert {}^2D_5(q^2)\rvert\lvert A_1(q^2)\rvert$	q-1	2|q-1
		$\lvert {}^2D_5(q^2)\rvert\lvert A_1(q)\rvert^2$	q+1	2|q-1
		$\lvert D_5(q)\rvert\lvert A_1(q^2)\rvert$	q+1	2|q-1
D_5+A_2	Z_2	$\lvert D_5(q)\rvert\lvert A_2(q)\rvert$	q-1	
		$\lvert {}^2D_5(q^2)\rvert\lvert {}^2A_2(q^2)\rvert$	q+1	
D_6+A_1	Z_2	$\lvert D_6(q)\rvert\lvert A_1(q)\rvert$	q-1	2|q-1
			q+1	2 q-1
E_6+A_1	Z_2	$\lvert E_6(q)\rvert\lvert A_1(q)\rvert$	q-1	
		$\lvert {}^2E_6(q^2)\rvert\lvert A_1(q)\rvert$	q+1	
D_7	Z_2	$\lvert D_7(q)\rvert$	q-1	
		$\lvert {}^2D_7(q^2)\rvert$	q+1	
E_7	Z_2	$\lvert E_7(q)\rvert$	q-1	
			q+1	
$2A_4$	Z_4	$\lvert A_4(q)\rvert^2$	1	5|q-1
		$\lvert {}^2A_4(q^2)\rvert^2$	1	5|q-4
		$\lvert {}^2A_4(q^4)\rvert$	1	5|q-3
			1	5|q-2
$A_5+A_2+A_1$	Z_2	$\lvert A_5(q)\rvert\lvert A_2(q)\rvert\lvert A_1(q)\rvert$	1	6|q-1
		$\lvert {}^2A_5(q^2)\rvert {}^2A_2(q^2)\rvert\lvert A_1(q)\rvert$	1	6|q+1

TABLE 6 (continued)

Type of Δ_J	Ω_J	$\lvert(M^g)_\sigma\rvert$	$\lvert(S^g)_\sigma\rvert$	condition for occurrence
A_7+A_1	Z_2	$\lvert A_7(q)\rvert\lvert A_1(q)\rvert$	1	$4\mid q-1$
		$\lvert {}^2A_7(q^2)\rvert\lvert A_1(q)\rvert$	1	$4\mid q+1$
A_8	Z_2	$\lvert A_8(q)\rvert$	1	$3\mid q-1$
		$\lvert {}^2A_8(q^2)\rvert$	1	$3\mid q+1$
D_5+A_3	Z_2	$\lvert D_5(q)\rvert\lvert A_3(q)\rvert$	1	$4\mid q-1$
		$\lvert {}^2D_5(q^2)\rvert {}^2A_3(q^2)\rvert$	1	$4\mid q+1$
D_8	1	$\lvert D_8(q)\rvert$	1	$2\mid q-1$
E_6+A_2	Z_2	$\lvert E_6(q)\rvert\lvert A_2(q)\rvert$	1	$3\mid q-1$
		$\lvert {}^2E_6(q^2)\rvert\lvert {}^2A_2(q^2)\rvert$	1	$3\mid q+1$
E_7+A_1	1	$\lvert E_7(q)\rvert\lvert A_1(q)\rvert$	1	$2\mid q-1$
E_8	1	$\lvert E_8(q)\rvert$	1	

Note: Table 2 was previously obtained by Chang and Ree [8]. While our own work was in progress we learned that Table 3 appears in Shoji's paper [13]. We have very recently seen our Table 4 in an unpublished paper by K. Mizuno. Our own methods are different from those used by Chang and Ree, Shoji and Mizuno.

CHAPTER 6

THE BRAUER COMPLEX AND THE NUMBER OF σ-STABLE SEMI-SIMPLE CONJUGACY CLASSES IN A GIVEN FAMILY

In this chapter G will denote a simple simply connected algebraic group. So in this case we have a one-to-one correspondence between the semi-simple conjugacy classes of G and the p'-rational points in the simplex $\overline{C}_o$.

Let s be a semi-simple element in the finite group G_σ and let $\overline{C_G(s)}$ be the G_σ-orbit determined by s, that is, the set of the σ-stable centralizers of semi-simple elements which are G_σ-conjugate to $C_G(s)$. In particular, all the elements in $[s]_{G_\sigma}$ give rise to the same orbit.

For the representation theory of the dual group G_σ^* of G_σ one is always interested to know how many semi-simple conjugacy classes give rise to the same orbit. For, each semi-simple irreducible representation constructed in [9] for the group G_σ^* arises from a semi-simple element in the group G_σ. As has been mentioned at the beginning of Chapter 4 these representations fall into families each of which corresponds to an orbit. The number of semi-simple irreducible representations belonging to a given family is equal to the number of semi-simple classes of G_σ which give rise to the orbit which corresponds to the family.

Using the Brauer complex we shall compute this number for the group of rank ≤ 2. This will suggest that the Brauer complex can be used to solve this problem for groups of higher ranks.

Let x_o be a σ-invariant point in $\overline{C}_o$ and let $\overline{\mathcal{Q}}_{\omega,y}$ be the alcove containing x_o. By definition x_o corresponds to a unique σ-stable semi-simple class $[s]_G$. We have seen that the orbit $\overline{C_G(s)}$ is parametrized by the family $(E,[\omega])$, where E is the type of the subset of roots in $\tilde{\Delta}$ which defines the face of $\overline{C}_o$ (equivalently of $\mathcal{Q}_{\omega,y}$) on which the σ-invariant point x_o lies. If another semi-simple class $[s']_G$ gives rise to the same orbit $\overline{C_G(s)}$, and if x'_o is the corresponding σ-invariant point lying in an alcove $\overline{\mathcal{Q}}_{\omega',y'}$, then we know that the faces on which x_o, x'_o lie are determined by sets of roots which are transformed to each other by an element τ of W. Moreover, $\tau\omega'\tau^{-1} \in [\omega]_{\Omega_J}$. For, let Δ_J, Δ'_J be the sets of roots which define these faces and let $t_o = h(\chi_{x_o})$, $t'_o = h(\chi_{x'_o})$ be the elements in the maximal torus T_o which correspond to the points x_o and x'_o respectively. Then Proposition 4.1 says that $C_G(t_o)^g = C_G(s)$, $C_G(t'_o)^{g'} = C_G(s')$, for some $g, g' \in G$, such that $\omega = \pi\,(g\sigma(g)^{-1}) \in \Omega_J$ and $\omega' = \pi(g'\sigma(g')^{-1}) \in \Omega_{J'}$. Now since we are dealing with Chevalley groups we may take the representative element n_τ of τ to be σ-stable. As $\tau(\Delta_J) = \Delta_{J'}$, we have $C_G(t_o)^{n_\tau} = C_G(t'_o)$. Thus the element $\omega'' = (n_\tau g'\sigma(g')^{-1}n_\tau^{-1}) = \tau\omega'\tau^{-1}$ is in $[\omega]_{\Omega_J}$ as required. If we work backwards we can see that if $\tau(\Delta_J) = \Delta_{J'}$ and $\tau\omega'\tau^{-1} \in [\omega]_J$, for some $\tau \in W$, then the two pairs $(\Delta_J,[\omega]_{\Omega_J})$, $(\Delta_{J'},[\omega']_{\Omega_{J'}})$ define the same orbit of centralizers of semi-simple elements in G_σ. Whenever this happens we shall say that these two pairs are W-conjugate.

Now for the alcoves of the Brauer complex we make the following definition.

Definition 6.1.

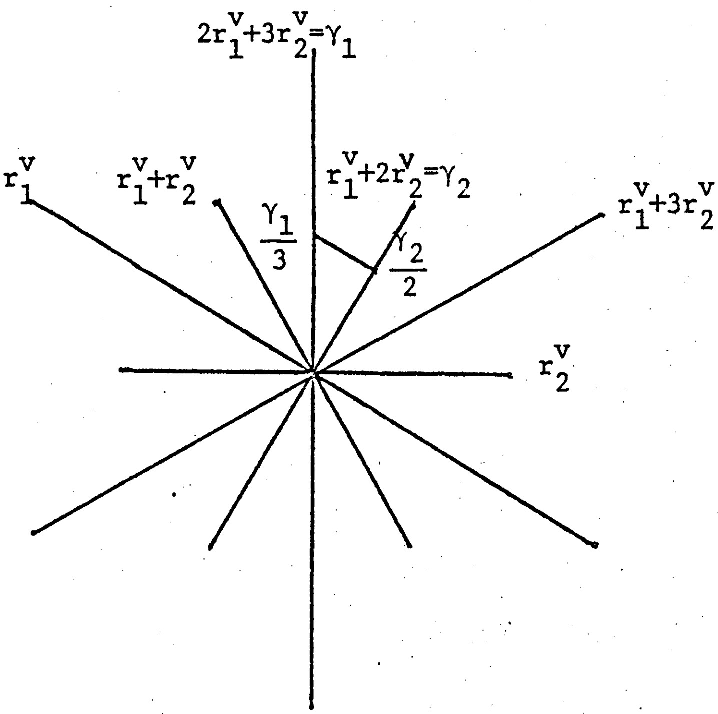
We say that an alcove of the Brauer complex is of type Δ_J if the σ-invariant point which is contained in its closure lies on the face of C_o of type J. If $\mathfrak{A}_{\omega,y}$ is an alcove of type Δ_J, then we shall say that this alcove is associated with the pair $(\Delta_J,[\omega]_{\Omega_J})$.

Thus the number of σ-stable semi-simple conjugacy classes which give rise to the given orbit of centralizers, is equal to the number of alcoves whose pair $(\Delta_J,[\omega]_{\Omega_J})$ determines the given orbit of centralizers. So instead of using algebraic methods, i.e. to work inside the group G_σ itself, we can use the geometry of the Brauer complex which gives an alternative way to find the number of semi-simple classes of G_σ which give rise to the given family

The number of semi-simple irreducible representations which belong to a given family for the groups of rank ≤ 2 have been determined using algebraic methods by various authors (e.g. for the group $G_2(q)$ see [8]). As described above, we can now determine this number by computing the number of alcoves (equivalently of σ-invariant points in $\overline{C}_o$) of the Brauer complex which give rise to a given family. We shall do this in detail for the group of type G_2 and write down in tables the numbers for the other groups of rank ≤ 2.

Type G_2.

The Weyl group $W = W(G_2)$ is isomorphic to the dihedral group of order 12 and is usually represented as a reflection group acting on:



where the indicated vectors are the positive coroots and γ_1, γ_2 are the fundamental coweights.

From the extended Dynkin diagram $\overset{-r_o}{\circ}$——$\overset{r_2}{\circ}\Rrightarrow\overset{r_1}{\circ}$ we obtain the following subsets: $\Delta_1 = \{r_1, r_2\}$, $\Delta_2 = \{-r_o, r_1\}$, $\Delta_3 = \{-r_o, r_2\}$, $\Delta_4 = \{r_1\}$, $\Delta_5 = \{r_2\}$, $\Delta_6 = \{-r_o\}$ and $\Delta_7 = \emptyset$. These are respectively of type G_2, $A_1 + \tilde{A}_1$, A_2, $\tilde{A}_1$, A_1, A_1 and $\emptyset$. Obviously we have just only one alcove of type Δ_1, this is the alcove $\mathcal{A}_o$. Now since the vertex of C_o defined by Δ_2 is the point $\gamma_{2/2}$ and since $\gamma_2 \in Y$ we see that if $2|q-1$, then there is only one alcove of type Δ_2, namely the alcove $\mathcal{A}_{1,\frac{q-1}{2}\gamma_2}$. If $2|q$, then there is no alcove of this type as in this case the vertex $\gamma_{2/2}$ is not a p'-rational point. Similarly, the vertex of C_o determined by Δ_3 is the point $\gamma_{1/3}$ and since $\gamma_1 \in Y$ we see that if $3|q-1$ or $3|q+1$, then there is only one alcove of type Δ_3, otherwise there is no alcove of this type. In particular, if $3|q-1$, then the alcove of type Δ_3 is associated with the pair $(\Delta_3,1)$ and is the alcove $\mathcal{O}_{1,\frac{q-1}{3}\gamma_1}$. If $3|q+1$, then the alcove of type Δ_3 is associated with the pair $(\Delta_3,\omega_2\omega_1\omega_2)$ and is the alcove $\mathcal{O}_{\omega_2\omega_1\omega_2,\ \frac{q-2}{3}\gamma_1+\gamma_2}$. Notice that $\Omega_3 = \{1,\omega_2\omega_1\omega_2\}$.

Now we consider the alcoves of type Δ_4. These are divided into alcoves which are associated with the pair $(\Delta_4,1)$ and into alcoves which are associated with the pair $(\Delta_4,\omega_2(\omega_1\omega_2)^2)$, as we have $\Omega_4 = \{1,\omega_2(\omega_1\omega_2)^2\}$. The number of alcoves which are associated with the latter pair is equal to the number of points in $\frac{1}{q}Y$ which lie on the face of C_o defined by Δ_4. For, these points are the relative positions of these alcoves with respect to C_o, as defined in pp.30. If q is odd, these points are of the form $\frac{n}{q}\gamma_2$, where $1 \le n \le \frac{q-1}{2}$. If q is a power of 2, then these points are the points $\frac{n}{q}\gamma_2$, where $1 \le n \le \frac{q-2}{2}$. Thus the number of alcoves associated with the pair $(\Delta_4,\omega_2(\omega_1\omega_2)^2)$ is $\frac{q-1}{2}$ if q is odd, otherwise this number is q/2, since in this case the alcove $\mathcal{O}_{\omega_2(\omega_1\omega_2)^2,\ q\frac{\gamma_2}{2}}$ is of type Δ_4. In the same way we find that the number of alcoves associated with the pair $(\Delta_4,1)$ is $\frac{q-3}{2}$ if $2|q-1$ and this number is $\frac{q-2}{2}$ if $2|q$.

Now the alcoves of type Δ_5 and Δ_6 are divided into alcoves which are associated with the pairs $(\Delta_5,1)$, $(\Delta_6,1)$, $(\Delta_5,\omega_1(\omega_2\omega_1)^2)$, (Δ_6,ω_1), as $\Omega_5 = \{1,\omega_1(\omega_2,\omega_1)^2\}$ and $\Omega_6 = \{1,\omega_1\}$. The first two pairs are W-conjugate and they determine the family $(A_1,1)$.

So to obtain the number of alcoves which correspond to the family $(A_1,1)$ we have to add the number of alcoves which are associated with each of the pairs $(\Delta_5,1)$ and $(\Delta_6,1)$. In the same way as before we find here that, if $3|q+1$, for the first pair this number is $\frac{q-2}{3}$ and for the second pair is $\frac{q-5}{6}$. Thus the number of the semi-simple conjugacy classes which give rise to the same orbit of centralizers parametrized by the family $(A_1,1)$, is $\frac{q-3}{2}$. The same number we obtain if $3|q$. If $3|q-1$, we find that this number is $\frac{q-5}{2}$.

Similarly, the pairs $(\Delta_5,\omega_1(\omega_2\omega_1)^2)$ and (Δ_6,ω_1) are W-conjugate and we obtain the following numbers for the alcoves associated with each of these pairs:

$$(\Delta_5,\omega_1(\omega_2\omega_1)^2) \quad \begin{cases} \frac{q-2}{3} & \text{if } 3|q+1 \\ \frac{q-1}{3} & \text{if } 3|q-1 \\ \frac{q}{3} & \text{if } 3|q. \end{cases}$$

$$(\Delta_6,\omega_1) \quad \begin{cases} \frac{q-5}{6} & \text{if } 3|q+1 \\ \frac{q-1}{6} & \text{if } 3|q-1 \\ \frac{q-3}{6} & \text{if } 3|q. \end{cases}$$

Thus the number of semi-simple classes which correspond to the family determined by these pairs is as follows: $\frac{q-3}{2}$ if $3|q+1$, $\frac{q-1}{2}$ if $3|q-1$ and $\frac{q-1}{2}$ if $3|q$. (We have assumed in this case that q is odd. For q even, see tables below).

Finally, similar calculations show that for the families $(\emptyset,[\omega])$, $\omega \in W$, the numbers in question are as follows:

$$(\emptyset,1)\quad \begin{cases} \dfrac{q^2-8q+19}{12} & \text{if } 2\nmid q,\ 3|q-1 \\[2ex] \dfrac{q^2-8q+16}{12} & \text{if } 2|q,\ 3|q-1 \\[2ex] \dfrac{q^2-8q+15}{12} & \text{if } 2\nmid q,\ 3|q+1 \text{ or if } 3|q \\[2ex] \dfrac{q^2-8q+12}{12} & \text{if } 2|q,\ 3|q+1. \end{cases}$$

$$(\emptyset,-1)\quad \begin{cases} \dfrac{q^2-4q+3}{12} & \text{if } 2\nmid q,\ 3|q-1 \text{ or if } 3|q \\[2ex] \dfrac{q^2-4q}{12} & \text{if } 2|q,\ 3|q-1 \\[2ex] \dfrac{q^2-4q+7}{12} & \text{if } 2\nmid q,\ 3|q+1 \\[2ex] \dfrac{q^2-4q+4}{12} & \text{if } 2|q,\ 3|q+1 \end{cases}$$

$$(\emptyset,[\omega_1]) \quad \begin{cases} \frac{(q-1)^2}{4} & \text{if } 2\nmid q \text{ and } 3\nmid q \text{ or if } 3|q \\ \frac{q^2-2q}{4} & \text{if } 2|q. \end{cases}$$

$$(\emptyset,[\omega_2]) \quad \begin{cases} \frac{(q-1)^2}{4} & \text{if } 2\nmid q \text{ and } 3\nmid q \text{ or if } 3|q \\ \frac{q^2-2q}{4} & \text{if } 2|q. \end{cases}$$

$$(\emptyset,[(\omega_1\omega_2)^2]) \quad \begin{cases} \frac{q^2+q-2}{6} & \text{if } 3|q-1 \\ \frac{q^2+q}{6} & \text{if } 3|q+1 \text{ or if } 3|q. \end{cases}$$

$$(\emptyset,[\omega_1\omega_2]) \quad \begin{cases} \frac{q^2-q}{6} & \text{if } 3|q-1 \text{ or if } 3|q \\ \frac{q^2-q-2}{6} & \text{if } 3|q+1. \end{cases}$$

In concluding this chapter it is useful to have for all the groups G_σ of rank ≤ 2 the number of semi-simple classes (equivalently of alcoves of the Brauer complex) which give rise to a given family. These numbers are given in the following tables. In these tables ω_1, ω_2 denote as usual the reflections in the hyperplanes orthogonal to the simple roots.

Type A_1

Family	Number of semi-simple classes. $2\mid q-1$	$2\mid q$
$(A_1,1)$	2	1
$(\emptyset,1)$	$\frac{q-3}{2}$	$\frac{q-2}{2}$
$(\emptyset,\omega_1)$	$\frac{q-1}{2}$	$\frac{q}{2}$

Type A_2

Family	Number of semi-simple classes. $3\mid q-1$	$3\nmid q-1$
$(A_2,1)$	3	1
$(A_1,1)$	$q-4$	$q-2$
$(\emptyset,1)$	$\frac{q^2-5q+10}{6}$	$\frac{q^2-5q+6}{6}$
$(\emptyset,[\omega_1])$	$\frac{q(q-1)}{2}$	$\frac{q(q-1)}{2}$
$(\emptyset,[\omega_1\omega_2])$	$\frac{q^2+q-2}{3}$	$\frac{q(q+1)}{3}$

Type B_2

Family	Number of semi-simple classes $2\mid q-1$	$2\mid q$
$(B_2,1)$	2	1
$(2A_1,1)$	1	none
$(2A_1,\omega_2\omega_1\omega_2)$	none	none
$(A_1,1)$	$q-3$	$\frac{q-2}{2}$
$(A_1,\omega_2\omega_1\omega_2)$	$q-1$	$\frac{q}{2}$
$(\tilde{A}_1,1)$	$\frac{q-3}{2}$	$\frac{q-2}{2}$
$(\tilde{A}_1,\omega_1\omega_2\omega_1)$	$\frac{q-1}{2}$	$\frac{q}{2}$
$(\emptyset,1)$	$\frac{q^2-8q+15}{8}$	$\frac{q^2-6q+8}{8}$
$(\emptyset,-1)$	$\frac{q^2-4q+3}{8}$	$\frac{q^2-2q}{8}$
$(\emptyset,[\omega_1])$	$\frac{q^2-4q+3}{4}$	$\frac{q^2-2q}{4}$
$(\emptyset,[\omega_2])$	$\frac{(q-1)^2}{4}$	$\frac{q^2-2q}{4}$
$(\emptyset,[\omega_1\omega_2])$	$\frac{q^2-1}{4}$	$\frac{q^2}{4}$

Type G_2

Family	Number of semi-simple classes: $2\mid q-1$, $3\mid q-1$	$2\mid q-1$, $3\mid q+1$	$2\mid q-1$, $3\mid q$	$2\mid q$, $3\mid q-1$	$2\mid q$, $3\mid q+1$
$(G_2,1)$	1	1	1	1	1
$(A_2,1)$	1	none	none	1	none
$(A_2,\omega_2\omega_1\omega_2)$	none	1	none	none	1
$(A_1+\tilde{A}_1,1)$	1	1	1	none	none
$(A_1,1)$	$\frac{q-5}{2}$	$\frac{q-3}{2}$	$\frac{q-3}{2}$	$\frac{q-4}{2}$	$\frac{q-2}{2}$
(A_1,ω_1)	$\frac{q-1}{2}$	$\frac{q-3}{2}$	$\frac{q-1}{2}$	$\frac{q}{2}$	$\frac{q-2}{2}$
$(\tilde{A}_1,1)$	$\frac{q-3}{2}$	$\frac{q-3}{2}$	$\frac{q-3}{2}$	$\frac{q-2}{2}$	$\frac{q-2}{2}$
$(\tilde{A}_1,\omega_2(\omega_1\omega_2)^2)$	$\frac{q-1}{2}$	$\frac{q-1}{2}$	$\frac{q-1}{2}$	$q/2$	$\frac{q}{2}$
$(\emptyset,1)$	$\frac{q^2-8q+19}{12}$	$\frac{q^2-8q+15}{12}$	$\frac{q^2-8q+15}{12}$	$\frac{q^2-8q+16}{12}$	$\frac{q^2-8q+12}{12}$
$(\emptyset,-1)$	$\frac{q^2-4q+3}{12}$	$\frac{q^2-4q+7}{12}$	$\frac{q^2-4q+3}{12}$	$\frac{q^2-4q}{12}$	$\frac{q^2-4q+4}{12}$
$(\emptyset,[\omega_1])$	$\frac{(q-1)^2}{4}$	$\frac{(q-1)^2}{4}$	$\frac{(q-1)^2}{4}$	$\frac{q^2-2q}{4}$	$\frac{q^2-2q}{4}$
$(\emptyset,[\omega_2])$	$\frac{(q-1)^2}{4}$	$\frac{(q-1)^2}{4}$	$\frac{(q-1)^2}{4}$	$\frac{q^2-2q}{4}$	$\frac{q^2-2q}{4}$
$(\emptyset,[(\omega_1\omega_2)^2])$	$\frac{q^2+q-2}{6}$	$\frac{q(q+1)}{6}$	$\frac{q(q+1)}{6}$	$\frac{q^2+q-2}{6}$	$\frac{q(q+1)}{6}$
$(\emptyset,[\omega_1\omega_2])$	$\frac{q(q-1)}{6}$	$\frac{q^2-q-2}{6}$	$\frac{q(q-1)}{6}$	$\frac{q(q-1)}{6}$	$\frac{q^2-q-2}{6}$

Finally we make the following conjecture.

<u>Conjecture</u>:

Let G_σ be a finite Chevalley group of universal type of rank ℓ. Then except for the bad primes the number of semisimple classes which give rise to a given family $(\emptyset, [\omega])$ is $\frac{n(q)}{|C_W(\omega)|}$, $f_1(q) \leq n(q) \leq f_2(q)$ where $f_1(t)$, $f_2(t)$ are monic polynomials in $Z[t]$ of degree ℓ-[illegible] differing only in their constant terms.

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